Mechanics Research Communications 38 (2011) 382–387

Contents lists available at ScienceDirect

Mechanics Research Communications

journal homepage: www.elsevier.com/locate/mechrescom

A note on the viscous, 1D shallow water equations: Traveling wave phenomena

S.A. Chin-Bing, P.M. Jordan*, A. Warn-Varnas

U.S. Naval Research Laboratory, Stennis Space Center, MS 39529, USA

ARTICLE INFO

Article history:
Received 20 September 2010
Received in revised form 6 May 2011
Available online 19 May 2011

Keywords:
Viscous shallow water equations
Traveling wave solutions
Bistability
Abel equations
Viscoelasticity

ABSTRACT

Exact traveling wave solutions (TWS)s of the one-dimensional (1D) shallow water equations are derived and studied in the case of a viscous fluid. These TWSs, which satisfy special cases of Abel's equation, are shown to take the form of kinks, which are not classic Taylor shocks, and to admit bifurcations and steepening. Stability issues are also addressed, asymptotic/limiting case expressions are presented, the possibility of hysteresis is explored, and it is established that bistability only occurs for left-running waves. Last, it is shown that the free surface height is capable of behaving similarly to the strain exhibited by a class of nonlinear viscoelastic media.

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1. Introduction

In one spatial dimension, and assuming a flat, rigid channel bottom, the viscous shallow water equations, also known as the Saint-Venant equations, can be expressed as (Mascia and Rousset, 2007)

$$h_\tau + (hv)_\chi = 0, \qquad (hv)_\tau + (hv^2)_\chi + \frac{1}{2}g(h^2)_\chi = \nu(hv_\chi)_\chi, \tag{1}$$

where $h(>0)$ is the height of the free surface above the flat bottom, v is the velocity in the χ-direction, $g(\approx 9.8\,\mathrm{m/s^2})$ is the acceleration due to gravity, and $\nu(>0)$ is the kinematic viscosity of the fluid under consideration.

System (1) is derived from depth-integrating the Navier–Stokes equations, under the assumption that the horizontal length scale is much greater than the vertical length scale, and then invoking the hydrostatic approximation to eliminate the pressure. The equations of Saint-Venant arise in a wide range of applications, both practical and theoretical. It is not surprising, therefore, given how important the ability to predict and control the movement of water is to modern societies, that this system of nonlinear PDEs has been, and continues to be, the subject of intense study, over 160 years after it was first derived; see, e.g., Debnath (1994) and LeVeque (2002).

The present Note is devoted to an analytical study of the TWSs admitted by System (1). In particular, we establish that these waveforms take the form of (non-Taylor shock) kinks and that they suffer, mathematically speaking, the same bifurcations identified with the phenomenon of hysteresis. The findings presented here also indicate that if the kinks are slowly propagating, then h exhibits behavior similar to that of the strain created in a well known class of nonlinear viscoelastic media by the passage of a kink-type traveling wave.

2. Traveling wave analysis

To this end, we begin by introducing the dimensionless independent variables $x = \chi/L$ and $t = c_0\tau/L$, where the positive constant L denotes a characteristic length, and then set $v = c_0 V(\xi)$ and $h = h_e H(\xi)$, where V and H are the dimensionless dependent variables. Here, $\xi := x - ct$ is our wave variable; the nonzero constant c denotes the speed of the assumed traveling waveforms, where $c \gtrless 0$ respectively correspond to right- and left-running waves; and $c_0 = \sqrt{gh_e}$ is the small-amplitude wave speed, where the constant $h_e(>0)$ denotes the equilibrium value of h. On substituting these ansatzs into System (1), integrating each of the resulting ordinary differential equations (ODE)s once, and then solving for the ensuing constants of integration by imposing the equilibrium state conditions $V = 0$ and $H = 1$, we end up with

$$V = c(1 - H^{-1}) \quad \text{and} \quad \epsilon HV' = H^2 - 2cHV + 2HV^2 - 1, \tag{2}$$

where $H \in (0, \infty)$, we have set $\epsilon := 2c_0^{-1}\nu/L$ for convenience, and a prime denotes $d/d\xi$.

On eliminating V between the equations in (2) we obtain the following special case of Abel's equation (Murphy, 1960):

$$\epsilon c H' = H^3 - (2c^2 + 1)H + 2c^2, \tag{3}$$

* Corresponding author. Tel.: +1 228 688 4338; fax: +1 228 688 5049.
E-mail address: pedro.jordan@nrlssc.navy.mil (P.M. Jordan).

0093-6413/$ – see front matter. Published by Elsevier Ltd.
doi:10.1016/j.mechrescom.2011.05.003

while the elimination of H from (3) using $(2)_1$ results in the more complicated, but still separable, ODE

$$\epsilon c\left(1-\frac{V}{c}\right)V' = -2V\left[V^2+\left(\frac{1-4c^2}{2c}\right)V-(1-c^2)\right], \tag{4}$$

which is also a form of Abel's equation (again, see Murphy, 1960). Focusing our attention on the former, due to its relative simplicity, a stability analysis indicates that, counting multiplicities, (3) admits the following equilibrium solutions:

$$\bar{H}=\{H_1,H_2,H_3\},\quad \text{where } H_1=1,\ H_{2,3}=-\frac{1}{2}\left[1\mp\sqrt{1+8c^2}\right], \tag{5}$$

of which only the first two are physically relevant since H_3 is always negative. Here, we observe that the equilibrium $\bar{H}=H_1$ is stable (resp. unstable) for $c>1$ (resp. $c<1$).

Assuming $c>0$ in the remainder of this section, we observe the following. For $c=1$, (3) exhibits a bifurcation; i.e., $c=1$ is the bifurcation value of the wave speed. This means that as the value of c passes through one, the two equilibria $\bar{H}=\{H_1,H_2\}$ first coalesce at the former, at which point $\bar{H}=1$ becomes a double zero of the cubic on the right-hand side of (3), and they then switch their stability; see Fig. 1. Thus when $c=1$, $\bar{H}=1$ attracts only from one side, namely, the left, and is thus referred to as a semi-stable (Strogatz, 1994) equilibrium, although other authors use the terms nonhyperbolic equilibrium (Hale and Koçak, 1991) and saddle point (Bender and Orszag, 1999).

Having completed our stability analysis, we are now ready to derive the exact TWSs for H and V, where we observe that the cases $c\neq 1$ and $c=1$ must be handled separately. First, however, we impose the wavefront conditions $H(0)=H_0$ and $V(0)=V_0$, where H_0 is a positive constant and V_0 is related to H_0 via $(2)_1$, so that the resulting constants of integration can be determined.

2.1. Results for $c\neq 1$

Setting aside the equilibrium solutions for the moment, we return to (3) and (4), separate variables, integrate, and then enforce the wavefront conditions. This yields the exact, but implicit, solutions

$$\xi=\frac{\epsilon c}{4(1-c^2)}\left\{\left(\frac{6}{\sqrt{1+8c^2}}\right)\operatorname{Arctanh}\left(\frac{1+2\mathcal{H}}{\sqrt{1+8c^2}}\right)+\ln\left[\frac{(1-\mathcal{H})^2}{\mathcal{H}^2+\mathcal{H}-2c^2}\right]\right\}\Bigg|_{H_0}^{H}\quad(-\infty<\xi<\infty), \tag{6}$$

$$\xi=\frac{-\epsilon c}{4(1-c^2)}\left\{\left(\frac{6}{\sqrt{1+8c^2}}\right)\operatorname{Arctanh}\left[\frac{1-4c(c-\mathcal{V})}{\sqrt{1+8c^2}}\right]-\ln\left[\frac{\mathcal{V}^2}{2c\mathcal{V}^2+(1-4c^2)\mathcal{V}-2c(1-c^2)}\right]\right\}\Bigg|_{V_0}^{V}\quad(-\infty<\xi<\infty), \tag{7}$$

respectively, for H and V, where $\min(1, H_2) < H_0 < \max(1, H_2)$ is assumed here (recall Fig. 1).

In Figs. 2 and 3 we have plotted, for $c<1$ and $c>1$, respectively, the integral curves given in (6) and (7). In both of these figures it is clear that $H(\xi)>V(\xi)$, for all $\xi\in\mathbb{R}$, and that H is strictly positive, as System (1) dictates. Also, while not immediately evident from the expressions in (6) and (7), these solution profiles assume the form of kinks (Angulo, 2009, pp. 25–26), which exhibit a relatively high degree of symmetry.[1]

2.2. Results for $c=1$

In this case we are able to obtain exact, *explicit* results, namely,

$$H(\xi)=\begin{cases}\dfrac{-2+W_{-1}[q(H_0)\exp[q(H_0)]\exp(9\xi/\epsilon)]}{1+W_{-1}[q(H_0)\exp[q(H_0)]\exp(9\xi/\epsilon)]}, & H_0>1,\ \text{for } \xi\in(-\infty,\xi_\infty),\\ 1, & H_0=1,\ \text{for } \xi\in(-\infty,\infty),\\ \dfrac{-2+W_0[q(H_0)\exp[q(H_0)]\exp(9\xi/\epsilon)]}{1+W_0[q(H_0)\exp[q(H_0)]\exp(9\xi/\epsilon)]}, & H_0<1,\ \text{for } \xi\in(\xi_z,\infty);\end{cases} \tag{8}$$

$$V(\xi)=\begin{cases}\dfrac{3}{2-W_{-1}[f(V_0)\exp[f(V_0)]\exp(9\xi/\epsilon)]}, & V_0\in(0,1),\ \text{for } \xi\in(-\infty,\xi_\infty),\\ 0, & V_0=0,\ \text{for } \xi\in(-\infty,\infty),\\ \dfrac{3}{2-W_0[f(V_0)\exp[f(V_0)]\exp(9\xi/\epsilon)]}, & V_0<0,\ \text{for } \xi\in(\xi_z,\infty).\end{cases} \tag{9}$$

Here, $W_r(\cdot)$ denotes the rth branch of the Lambert W-function (see Appendix A); we have set $q(H_0):=(2+H_0)/(1-H_0)$ and $f(V_0):=2-3/V_0$ for convenience;

$$\xi_\infty:=\frac{\epsilon}{9}\left\{\frac{3-(H_0-1)\ln[(2+H_0)(H_0-1)^{-1}]}{H_0-1}\right\}\quad(H_0>1), \tag{10}$$

where $H(\xi)|_{c=1}\to\infty$ as $\xi\to\xi_\infty$ (from below); and

$$\xi_z:=-\frac{\epsilon}{9}\left\{\frac{3H_0+(1-H_0)\ln[(2+H_0)(2-2H_0)^{-1}]}{1-H_0}\right\}\quad(H_0<1), \tag{11}$$

where $H(\xi_z)|_{c=1}=0$.

Remark 1. An inspection of (8) and (9) reveals that the only physically realistic[2] solutions possible for this case are

$$H(\xi)=1\quad\text{and}\quad V(\xi)=0\quad(c=1), \tag{12}$$

i.e., the defining relations for the equilibrium state of the fluid.

3. Analytical results

While they are exact, the solutions appearing in (6) and (7) are, from the analytical standpoint, rather complicated, and thus do not immediately provide the physical insight we seek. Fortunately, however, approximate/asymptotic expressions, which are both simpler than their exact counterparts and explicit, can be derived from (6) and (7), as well as from their governing ODEs, by, e.g., expanding about points of physical significance.

[1] In the sense that the "tanh" TWS admitted by Burgers' equation, a particular type of kink known as a *Taylor shock*, can be made perfectly symmetric, while the profiles appearing in Figs. 1 and 3.8 respectively of Jordan (2006) and Rasmussen (2009) are clearly asymmetric.

[2] Specifically, solutions that are bounded, defined on the entire ξ-axis, and for which H is strictly positive; e.g., TWS in the form of kinks.

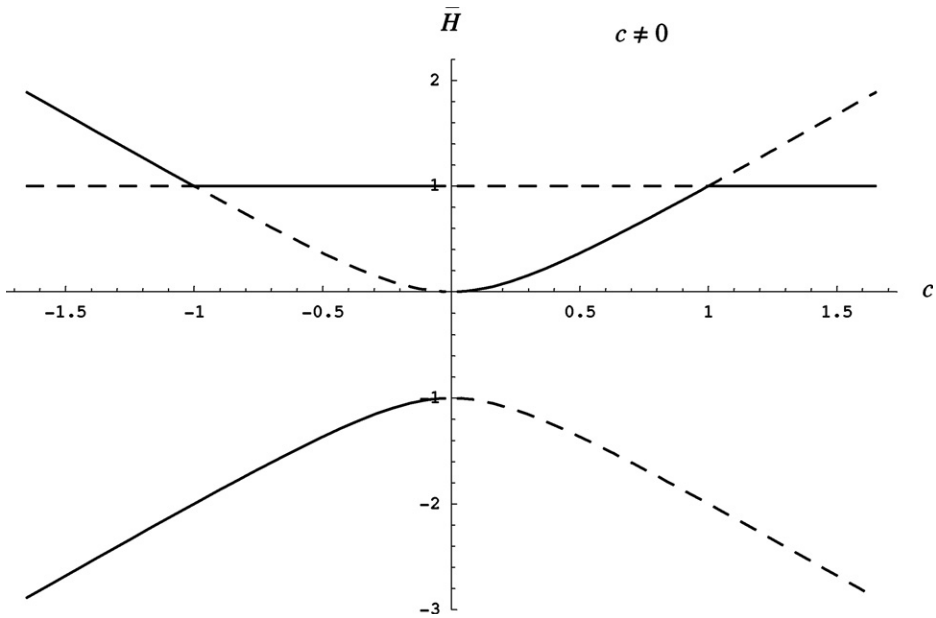

Fig. 1. $\bar{H}(= H_{1,2,3})$ vs. c, where $c \neq 0$ is assumed. The solid and broken branches correspond to the stable and unstable equilibria, respectively.

3.1. Small-$|\xi|$ approximations and shock thicknesses

Borrowing from the terminology of gas dynamics theory, we define the shock thicknesses, $\ell_{H,V}(>0)$, of our kink waveforms as

$$\ell_H := \frac{c\epsilon|1 - H_2|}{|H_0^3 - (2c^2 + 1)H_0 + 2c^2|},$$
$$\ell_V := \frac{\epsilon|(c - V_0)V_2|}{|2V_0^3 + c^{-1}(1 - 4c^2)V_0^2 - 2(1 - c^2)V_0|}, \tag{13}$$

where $V_2 = c(1 - 1/H_2)$.

Using (13) and elementary calculus, the following linear, small-$|\xi|$ approximations are readily constructed: $H(\xi) \approx H_0 - \xi/\ell_H$ and $V(\xi) \approx V_0 - \xi/\ell_V$, which are valid for $2|\xi| \ll \ell_H$ and $2|\xi| \ll \ell_V$, respectively. Unfortunately, however, while they are easy to derive and use, the intervals over which these simple expressions are valid are rather limited.

To obtain small-$|\xi|$ approximations with wider ranges of validity, we must return to our exact solutions. In the case of H, e.g., expanding (6) about $H = H_0$ yields, after simplifying and rearranging terms, the power series

$$\frac{\xi}{\epsilon c} = \frac{H_0 - H}{(1 - H_0)(H_0^2 + H_0 - 2c^2)} + \frac{(1 + 2c^2 - 3H_0^2)(H_0 - H)^2}{2(1 - H_0)^2(H_0^2 + H_0 - 2c^2)^2} + \mathcal{O}[(H_0 - H)^3]. \tag{14}$$

Thus, we see that (14) can be solved as a quadratic, cubic, etc., polynomial in $H_0 - H$ by simply neglecting the appropriate higher-order terms.

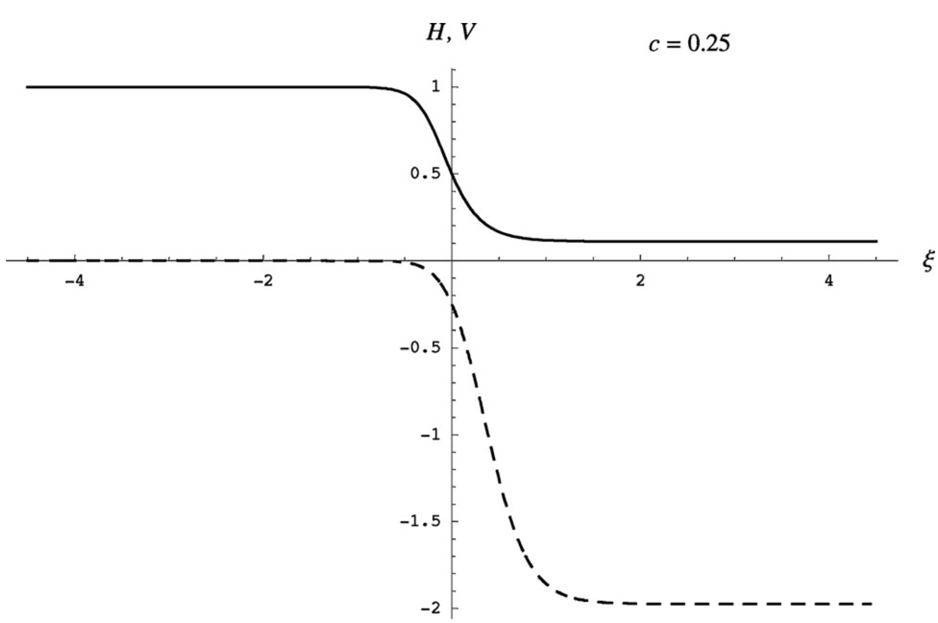

Fig. 2. H vs. ξ (solid) and V vs. ξ (broken) for $c = 0.25$ ($\Rightarrow H_2 \approx 0.112$), $H_0 = 0.50$, and $\epsilon = 1.0$. Here, the equilibria $\bar{H} = \{H_1, H_2\}$ are unstable and stable, respectively.

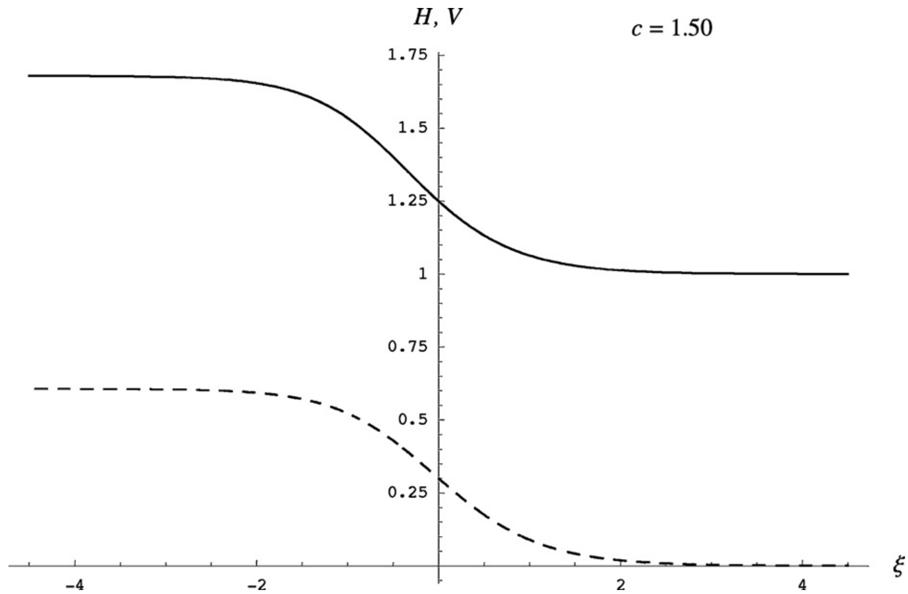

Fig. 3. H vs. ξ (solid) and V vs. ξ (broken) for $c = 1.50$ ($\Rightarrow H_2 \approx 1.679$), $H_0 = 1.25$, and $\epsilon = 1.0$. Here, the equilibria $\bar{H} = \{H_1, H_2\}$ are stable and unstable, respectively.

Remark 2. An inspection of (13) reveals, that just as in the case of a Taylor shock TWS, the H and V (kink) solution profiles form shocks as the (dimensionless) viscosity coefficient ϵ tends to zero, i.e., $\ell_{H,V} \to 0$ as $\epsilon \to 0$.

3.2. Asymptotic results

In this subsection we give two large-$|\xi|$ expressions. They are based on expansions of (6) and (4) about the fluid's equilibrium state, i.e., about $H = 1$ and $V = 0$, respectively, and their relatively simple structure makes them amiable to probing by analytical means.

To this end, we return to (6) and expand its RHS about $H = 1$. On neglecting terms of $\mathcal{O}[(1-H)^2]$ and then solving the resulting expression for H in terms of ξ, we find that as $\xi \to \pm\infty$,

$$H(\xi) \sim 1 - \frac{2(1-c^2)}{3} W_0 \left\{ \left(\frac{3\exp[G(H_0, c)]}{\sqrt{2|1-c^2|}} \right) \times \exp\left[\frac{2(1-c^2)\xi}{\epsilon c} \right] \right\}, \quad \text{for } c \gtrless 1, \tag{15}$$

respectively. Here, $W_0(\cdot)$ denotes the principal branch of the Lambert W-function (again, see Appendix A); $c > 0$ is assumed; and

$$G(H_0, c) = \left(\frac{3}{\sqrt{1+8c^2}} \right) \left[\text{Arctanh}\left(\frac{1+2H_0}{\sqrt{1+8c^2}} \right) - \text{Arctanh}\left(\frac{3}{\sqrt{1+8c^2}} \right) \right] + \ln\left(\frac{|1-H_0|}{\sqrt{|H_0^2 + H_0 - 2c^2|}} \right). \tag{16}$$

In contrast, dividing (4) through by $(1 - V/c)$ and then expanding the RHS under the assumption $|V| \ll c$ yields, after neglecting terms $\mathcal{O}(c^{-3}|V|^3)$, the following special case of Bernoulli's equation:

$$\epsilon V' \approx 2c^{-1}(1-c^2)V + c^{-2}(1+2c^2)V^2, \tag{17}$$

the exact solution of which is easily determined. Thus, it is readily seen that as $\xi \to \pm\infty$,

$$V(\xi) \simeq \frac{c^3 - c}{1+2c^2} \left\{ 1 - \tanh\left[\frac{(1-c^2)(\xi_0 - \xi)}{\epsilon c} \right] \right\}, \quad \text{for } c \gtrless 1, \tag{18}$$

respectively, where $0 < |1 - H_0| \ll 1$ is also assumed and

$$\xi_0 := \left(\frac{\epsilon c}{1-c^2} \right) \text{Arctanh}\left[1 - \frac{V_0(1+2c^2)}{c^3 - c} \right]. \tag{19}$$

While the expressions given in (15) and (18) correspond to the same asymptotic regimes, they are clearly different in terms of their analytical structure, with the former being the more complicated of the two. It should also be noted that, while (15) does *not* satisfy $H(0) = H_0$, (18) satisfies *exactly* the same wavefront condition as (7), i.e., $V(0) = V_0$.

Of course, the corresponding approximations about H_2 and V_2 can also be derived using the approach described above; however, in the interest of brevity, we leave this task to the reader.

Remark 3. If one is only interested in the extreme far-field regimes, then further simplification of (15) is possible using (A.3).

Remark 4. From (17) and (18) it is clear that, in the indicated regimes, the asymptotic behavior of V is described by a Taylor shock profile, and thus qualitatively similar to that of the TWS of Burgers' equation.

3.3. Approximations valid for small values of the wave speed

In the case of small-$|c|$, (3) and (4) are approximated by

$$\epsilon c H' + H - H^3 \simeq 0 \quad \text{and} \quad V\left[\epsilon V' - 2\left(V^2 + \frac{1}{2}c^{-1}V - 1 \right) \right] \simeq 0, \tag{20}$$

which, just like their exact counterparts, are satisfied by the equilibrium state conditions $H(\xi) = 1$ and $V(\xi) = 0$. Seeking instead only nontrivial solutions of the ODEs in (20), we separate variables and

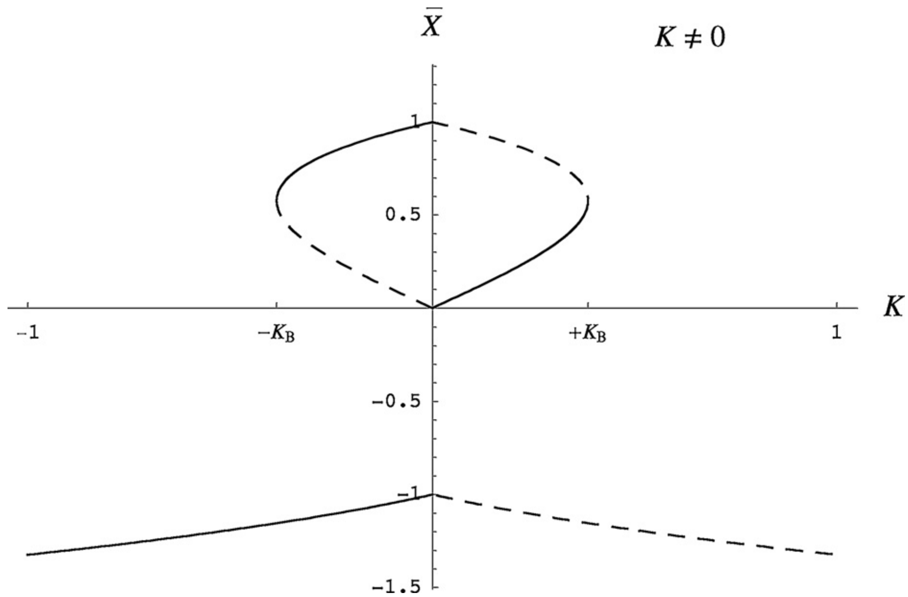

Fig. 4. $\bar{X}$ vs. K, where $K \neq 0$ is assumed. As in Fig. 1, the solid and broken branches correspond to the stable and unstable equilibria, respectively, and only the branches in the upper half-plane are physically relevant.

integrate. After enforcing our original wavefront conditions and then solving for H and V in terms of ξ, we find that for all $\xi \in \mathbb{R}$,

$$H(\xi) \simeq \frac{1}{\sqrt{1+(H_0^{-2}-1)\exp[2\xi/(c\epsilon)]}}, \tag{21}$$

$$V(\xi) \simeq -\frac{1}{4c}\left\{1-\sqrt{1+16c^2}\tanh\left[\frac{\sqrt{1+16c^2}(\xi_c-2\xi/\epsilon)}{4c}\right]\right\}, \tag{22}$$

provided $|c| \ll 1$ is sufficiently small. Here, $H_0 \in (H_2, 1)$ is assumed, where we observe that $H_2 \sim 2c^2(1-2c^2)$ as $c \to 0$, and

$$\xi_c := 4c(1+16c^2)^{-1/2}\,\text{Arctanh}\left[\frac{1+4cV_0}{(1+16c^2)^{1/2}}\right], \tag{23}$$

where it should be noted that $V_0 \neq 0$ since $V_0 = c(1-1/H_0)$.

Remark 5. Making the associations $H \mapsto f$ and $(c\epsilon) \mapsto \lambda/\sigma$, and taking $H_0 = 1/2$, our small-$|c|$ approximation for H is seen to be equivalent to Eq. 19 of Destrade et al. (2009), which is an *exact* TWS of the weakly nonlinear PDE known as the *modified* Burgers' equation (MBE); see also Enflo et al. (2006), Jordan and Puri (2005) and the references therein.

Remark 6. From (22) we once again find V exhibiting the character of a Taylor shock, but now in the small-$|c|$ regime; recall Remark 4. And while (22) is an exact solution of $(20)_2$, the assumption of small-$|c|$ allows us to simplify the former by replacing the sum "$1+16c^2$" with unity, in both (22) and (23), thus giving the expected asymptotic values $-\frac{1}{2c}$, 0.

3.4. Does System (1) exhibit hysteresis?

On pp. 30–32 of their classic text, Hale and Koçak (1991) present a simple example of *hysteresis*,[3] a term describing systems which exhibit a memory response, admitted by an ODE of the same form as (3). In this subsection we explore the possibility that, like its fellow Abel equation in (Hale and Koçak, 1991, Eq. (2.4)), our ODE in (3) also exhibits hysteretic-type behavior.

To this end, we introduce the quantities

$$X = \frac{H}{\sqrt{1+2c^2}}, \qquad \zeta = \left(\frac{1+2c^2}{\epsilon|c|}\right)\xi,$$
$$K = \frac{2c^2\ \text{sgn}(c)}{(1+2c^2)^{3/2}} \quad (c \neq 0), \tag{24}$$

in terms of which (3) becomes

$$\dot{X} = \begin{cases} K+X-X^3, & K<0 \\ K-X+X^3, & K>0 \end{cases} \quad (K \neq 0). \tag{25}$$

Here, sgn(·) is the sign function, where we observe that sgn(K) = sgn(c); a superposed dot denotes d/dζ; and x, t, and c in (Hale and Koçak, 1991, Exp. (2.4)) correspond to X, ζ, and K, respectively, in (24).

Clearly, $K = \pm K_B$, where $K_B := 3/\sqrt{27}$, are the bifurcations values of (25) *and* (Hale and Koçak, 1991 Eq. (2.4)); however, as a comparison of Fig. 4 with (Hale and Koçak, 1991, Fig. 2.5) reveals, the corresponding bifurcation diagrams are *not* the same. In particular, we observe that while the latter figure depicts a strictly bistable system, the former shows that (25) is bistable (resp. monostable) for $K<0$ (resp. $K>0$).

Thus, while the possibility of (3) exhibiting hysteresis is plausible at first glance, based on (Hale and Koçak, 1991, Exp. (2.4)), the lack of bistability for $K>0$ (not to mention the restriction $K \neq 0$ and the physical requirement $X>0$) immediately rules out this phenomenon.

Remark 7. From the mathematical standpoint, the $K<0$ case of (25) is an example of bistability *without* hysteresis. Actually, one can find in the literature rather simple systems that are capable of exhibiting bistability with and without hysteresis, e.g., the Schlögl reaction (Guidi and Goldbeter, 1997), a tri-molecular process described by an ODE similar to (3).

[3] While the term hysteresis is most often associated with magnetic materials, its use is not exclusive to that field; see, e.g., Murray (1993) and Guidi and Goldbeter (1997), wherein several examples from biology and chemistry, respectively, are discussed.

4. Discussion

It is noteworthy that the MBE is the equation of motion for transverse waves in viscoelastic media described by the cubically perturbed Kelvin–Voigt model (see Enflo et al., 2006; Jordan and Puri, 2005 and the references therein). Given this fact, and the observation recorded in Remark 5, it seems reasonable, therefore, to expect H to behave much like the strain induced in a sample of such material, examples of which include organic polymers, rubber, and wood,[4] by the presence of a traveling waveform, provided $|c|$ is sufficiently small.

This finding is all the more interesting given the mathematical structure exhibited by the equations of Camassa–Holm (CH)[5] theory; specifically, the similarity of these PDEs to the equation of motion for second-grade fluids, which are a particular class of non-Newtonian fluids that exhibit a retarded stress response (Cioranescu and Girault, 1997). Naturally, it is of interest to determine if the viscous generalizations of other water wave models (see, e.g., Camassa et al., 1994; Christov, 2001; Dias and Kharif, 1999; Johnson, 2003 and the references therein) are capable of predicting non-Newtonian behavior, and if so, to identify the apparent rheology expressed.

And finally, it must be stressed that what has been presented here are only predictions—ones based solely on System (1). It is, therefore, important for us to be aware of the range of pitfalls that might arise when attempting to extrapolate our findings, i.e., findings derived from a *1D* model of fluid flow, to higher dimensions; see Escudero (2006) and the references therein. In the end, confirming/refuting what the present mathematical analysis has revealed can, of course, only be accomplished through the efforts of experimentalist.

Acknowledgements

This study was initiated by P.M.J. in March 2007 while he was a visitor at OCIAM, Mathematical Institute, Oxford University, Oxford, UK. The authors are indebted to the three anonymous reviewers for their valuable comments and suggestions. All figures appearing in this article were produced using the software package Mathematica (Version 5.2). This work was supported by ONR/NRL funding.

Appendix A. The Lambert *W*-function

The Lambert W-function is defined as the solution of the transcendental equation (Corless et al., 1996)

$$W(\zeta)\,\exp[W(\zeta)] = \zeta, \tag{A.1}$$

where ζ here denotes a complex quantity and the roots of this equation correspond to the branches of W. By convention, the rth branch of W is denoted as $W_r(\cdot)$, where $r = 0, \pm 1, \pm 2, \ldots$, with $W_0(\cdot)$ known as the *principal* branch.

Strictly speaking, however, W is *not* a function since there are *two* distinct real values of $W(x)$, which correspond to $r = 0, -1$, for every $x \in (-e^{-1}, 0)$, where we observe that $W_{-1}(x) < -1 < W_0(x) < 0$ for $x \in (-e^{-1}, 0)$. These two branches, which are the only ones to take on real values, coincide at the branch point $x = -e^{-1}$, where $W_0(-e^{-1}) = W_{-1}(-e^{-1}) = -1$. However, both $W_0(x)$ and $W_{-1}(x)$ are complex-valued for $x < -e^{-1}$.

And finally, with regard to the principal branch, we observe that W_0 admits the following large- and small-x asymptotic expressions (Corless et al., 1996):

$$W_0(x) \sim \ln(x) - \ln[\ln(x)] + \cdots \quad (x \to \infty), \tag{A.2}$$

$$W_0(x) \sim x - x^2 + \cdots \quad (x \to 0), \tag{A.3}$$

and where it should be noted that $W_0(x) \geq 0$ for $x \geq 0$.

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[4] See the URL: http://en.wikipedia.org/wiki/Viscoelasticity, last accessed by the authors on 4 May 2011.

[5] In the literature, the *viscous* CH equation is another name for the Navier–Stokes-alpha (NS-α) model (Foias et al., 2001); *the* CH equation refers to the lossless, 1D PDE put forth in 1993 as a model of dispersive shallow water waves (see Camassa et al., 1994 and the references therein).