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Superconvergence of discontinuous Galerkin solutions for a nonlinear scalar hyperbolic problem

Slimane Adjerid [a,*], Thomas C. Massey [b]

[a] Department of Mathematics, Virginia Polytechnic Institute and State University, 460 MCBryde Hall, Blacksburg, VA 24061, USA
[b] Naval Research Laboratory, Oceanography Division (7322), Stennis Space Center, MS 39529, USA

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Abstract

In this paper we study the superconvergence of the discontinuous Galerkin solutions for nonlinear hyperbolic partial differential equations. On the first inflow element we prove that the p-degree discontinuous finite element solution converges at Radau points with an $O(h^{p+2})$ rate. We further show that the solution flux converges on average at $O(h^{2p+2})$ on element outflow boundary when no reaction terms are present. For reaction–convection problems we establish an $O(h^{\min(2p+2,p+4)})$ superconvergence rate of the flux on element outflow boundary. Globally, we prove that the flux converges at $O(h^{2p+1})$ on average at the outflow of smooth-solution regions for nonlinear conservation laws. Numerical computations indicate that our results extend to nonrectangular meshes and nonuniform polynomial degrees. We further include a numerical example which shows that discontinuous solutions are superconvergent to the unique entropy solution away from shock discontinuities.

MSC: 65M60; 65M25; 65M15

Keywords: Discontinuous finite element methods; Superconvergence; Hyperbolic differential equations; Conservation laws

1. Introduction

The discontinuous Galerkin (DG) finite element method has been used to solve first-order hyperbolic problems and is gaining in popularity. The DG method was first used for the neutron equation [24]. Since then, DG methods have been used to solve hyperbolic [7–10,15,14,16,20], parabolic [17,18], and elliptic

* Corresponding author. Tel.: +1 540 231 5945; fax: +1 540 231 5960.
E-mail address: adjerids@math.vt.edu (S. Adjerid).

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[6,5,25] partial differential equations. For a more complete list of citations on the DG methods and its applications consult [13]. A main advantage of using discontinuous finite element basis is to simplify adaptive p- and h-refinement with hanging nodes.

High-order, $p > 0$, DG solutions for nonlinear hyperbolic problems exhibit spurious oscillations near discontinuities. These oscillations may be reduced by using either limiting [10,11] or shock capturing [12,21] techniques that force the DG solution to converge to the unique entropy solution under mesh refinement. Many techniques to suppress spurious oscillations have been suggested but none is totally successful. From previous computational experience [1], we discovered that limiting can reduce spurious oscillations near shock discontinuities but does not enhance superconvergence properties of the DG solution near shocks. For these reasons, we restrict our superconvergence error analysis to the local error behavior in smooth-solution regions.

Recently, Adjerid et al. [1] proved that smooth DG solutions of one-dimensional linear and nonlinear hyperbolic problems using p-degree polynomial approximations exhibit an $O(h^{p+2})$ superconvergence rate at the roots of Radau polynomial of degree $p + 1$. They used this result to construct asymptotically correct a posteriori error estimates. They further established a strong $O(h^{2p+1})$ superconvergence at the downwind end of every element. Krivodonova and Flaherty [22] proved a superconvergence result on average on the outflow edge of every element of unstructured triangular meshes and constructed a posteriori error estimates that converge to the true error under mesh refinement. Adjerid and Massey [4] extended these results for multi-dimensional problems using rectangular meshes and presented an error analysis for linear problems and problems with a nonlinear reaction term. They showed that the leading term in the true local error is spanned by two $(p + 1)$-degree Radau polynomials in the x- and y-directions, respectively. They further showed that a p-degree discontinuous finite element solution exhibits an $O(h^{p+2})$ superconvergence at Radau points obtained as a tensor product of the roots of Radau polynomial of degree $p + 1$. For a linear model problem they established that, locally, the solution flux is $O(h^{2p+2})$ superconvergent on average on the outflow element boundary and the global solution flux converges at an $O(h^{2p+1})$ rate on average at the outflow boundary of the domain. They used these superconvergence results to construct asymptotically exact a posteriori error estimates for linear and nonlinear hyperbolic problems. In this paper, we extend the error analysis of Adjerid and Massey [4] to nonlinear hyperbolic scalar problems of the form

$$\nabla \cdot \mathbf{F}(u) = h(x, y), \quad (x, y) \in \Omega = [0, 1]^2 \tag{1.1}$$

and

$$\nabla \cdot \mathbf{F}(u) + \phi(u) = h(x, y), \quad (x, y) \in \Omega = [0, 1]^2, \tag{1.2}$$

with boundary conditions

$$u|_{\partial\Omega_{\text{in}}} = g. \tag{1.3}$$

The inflow and outflow boundaries are defined as

$$\partial\Omega_{\text{in}} = \left\{ (x, y) \in \partial\Omega \left| \frac{\mathrm{d}\mathbf{F}}{\mathrm{d}u} \cdot \boldsymbol{v} \leqslant 0 \right. \right\} \tag{1.4a}$$

and

$$\partial\Omega_{\text{out}} = \left\{ (x, y) \in \partial\Omega \left| \frac{\mathrm{d}\mathbf{F}}{\mathrm{d}u} \cdot \boldsymbol{v} > 0 \right. \right\}, \tag{1.4b}$$

where the boundary of Ω, $\partial\Omega = \partial\Omega_{\text{in}} \cup \partial\Omega_{\text{out}}$ and $\boldsymbol{v}$ is the outward unit normal to $\partial\Omega$. The difficulty with nonlinear conservation laws (1.1) is that in general for smooth flux function $F(x, y)$ and smooth boundary conditions g, smooth solutions do not in general exist for all $(x, y) \in \Omega$. Thus, only weak solutions can be defined. Furthermore, a weak solution is unique if it satisfies the entropy condition [19].

In order to perform an error analysis on the first inflow element, we assume $\mathbf{F}:\mathbf{R}\to\mathbf{R}^2$, $\phi:\mathbf{R}\to\mathbf{R}$, $u:\mathbf{R}^2\to\mathbf{R}$, h and g to be analytic functions. On the first inflow element we show that the DG solution of (1.1) is $\mathrm{O}(h^{p+2})$ superconvergent at Radau points and the leading term in the error is a linear combination of two Radau polynomials. Moreover, the flux is $\mathrm{O}(h^{2p+2})$ superconvergent on average on the outflow boundary of the first inflow element. For reaction problems (1.2) the flux is $\mathrm{O}(h^{\min(2p+2,p+4)})$ superconvergent on the outflow boundary of the first inflow element.

If we further assume that $u(x,y)$ is smooth on $\tilde{\Omega}$ such that

$$\tilde{\Omega}\subset\Omega \quad \text{and} \quad \partial\tilde{\Omega}_{\text{in}}\subset\partial\Omega_{\text{in}}, \tag{1.5}$$

then the flux is $\mathrm{O}(h^{2p+1})$ superconvergent on average at the outflow boundary $\partial\tilde{\Omega}_{\text{out}}$.

Finally, computational results for a problem with a shock discontinuity reveal that similar local superconvergence results hold in smooth-solution regions away from the shock. Thus, the error in smooth-solution regions propagates at a higher order.

This paper is organized as follows: In Section 2 we state and prove the main superconvergence results. In Section 3 we show numerical results for several problems. We conclude with a few remarks.

2. Error analysis

In this section we will analyze the DG discretization error and show that the leading term in the error is proportional to $(p+1)$-degree Radau polynomials in the x- and y-directions. Prior to proving this result we need to recall a few preliminary lemmas.

The weak discontinuous Galerkin formulation is obtained by partitioning the domain Ω into $N=n\times n$ square elements and starting the integration with elements whose inflow boundary is on the domain inflow boundary.

In order to perform an error analysis we consider the first element $\Delta=[0,h]^2$ where $h=1/n$ and the space $\mathscr{V}_p$ of polynomial functions such that

$$\mathscr{P}_{p+1}\subset\mathscr{V}_p\cup\{x^{p+1},y^{p+1}\}, \quad p\geqslant 0, \tag{2.1a}$$

where $\mathscr{P}_k$ is the space of polynomials of degree k

$$\mathscr{P}_k=\left\{q|q=\sum_{m=0}^{k}\sum_{i=0}^{m}c_i^m x^i y^{m-i}\right\}. \tag{2.1b}$$

These spaces are suboptimal but they lead to a very simple a posteriori error estimator. For efficiency reasons we consider the smallest spaces that satisfy (2.1)

$$\mathscr{V}_p=\left\{V|V=\sum_{k=0}^{p}\sum_{i=0}^{k}c_i^k x^i y^{k-i}+\sum_{i=1}^{p}c_i^{p+1}x^i y^{p+1-i}\right\}. \tag{2.2}$$

We note that tensor product elements satisfy (2.1).

Assuming $\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}u}(u(0,0))=[\alpha_1,\alpha_2]^t$, with $\alpha_i>0$, $i=1,2$, one can prove that for h small enough the inflow boundary of Δ is $\Gamma_{\text{in}}=\Gamma_1\cup\Gamma_4$ where $\Gamma_1=\{(x,0),0<x<h\}$ and $\Gamma_4=\{(0,y),0<y<h\}$. The outflow boundary $\Gamma_{\text{out}}=\Gamma_2\cup\Gamma_3$ with $\Gamma_2=\{(h,y),0<y<h\}$ and $\Gamma_3=\{(x,h),0<x<h\}$.

The discontinuous Galerkin method for (1.1) consists of determining $U(x,y)\in\mathscr{V}_p$ on Δ such that

$$\int_{\Gamma_{\text{in}}}\boldsymbol{v}\cdot(\mathbf{F}(U^-)-\mathbf{F}(U))V\,\mathrm{d}\sigma+\int\int_{\Delta}[\nabla\cdot\mathbf{F}(U)-h(x,y)]V\,\mathrm{d}x\,\mathrm{d}y=0, \quad \forall V\in\mathscr{V}_p. \tag{2.3}$$

The boundary data U^- on Γ_{in} is

$$U^-(x,y) = \begin{cases} \pi g, & \text{if } (x,y) \in \Gamma_1, \\ \pi g, & \text{if } (x,y) \in \Gamma_4, \end{cases} \tag{2.4}$$

where πw is the p-degree polynomial that interpolates w at the roots of $(p+1)$-degree right Radau polynomial

$$R^+_{p+1}(\xi) = L_{p+1}(\xi) - L_p(\xi), \quad -1 \leqslant \xi \leqslant 1, \tag{2.5}$$

with L_p being Legendre polynomial of degree p.

Once we determine the solution on the first element Δ we proceed to the elements whose inflow boundaries are either on the inflow boundary of Ω or an outflow boundary of Δ and continue this process until the solution is determined in the whole domain. On an element whose inflow boundary is not on the boundary of Ω, U^- is defined as

$$U^-(x,y) = \lim_{s\to 0^+} U((x,y) + s\boldsymbol{v}), \quad (x,y) \in \Gamma_{\text{in}}. \tag{2.6}$$

The discontinuous Galerkin solution satisfies the DG orthogonality condition which is obtained by multiplying (1.1) by $V \in \mathscr{V}_p$, integrating over the element Δ and applying Green's formula to obtain

$$\int_\Gamma \boldsymbol{v} \cdot \mathbf{F}(u) V \,\mathrm{d}\sigma - \int\int_\Delta \mathbf{F}(u) \cdot \nabla V \,\mathrm{d}x\,\mathrm{d}y = \int\int_\Delta h(x,y) V \,\mathrm{d}x\,\mathrm{d}y. \tag{2.7}$$

Applying Green's formula to (2.3) yields

$$\int_{\Gamma_{\text{in}}} \boldsymbol{v} \cdot \mathbf{F}(U^-) V \,\mathrm{d}\sigma + \int_{\Gamma_{\text{out}}} \boldsymbol{v} \cdot \mathbf{F}(U) V \,\mathrm{d}\sigma - \int\int_\Delta \mathbf{F}(U) \cdot \nabla V \,\mathrm{d}x\,\mathrm{d}y = \int\int_\Delta h(x,y) V \,\mathrm{d}x\,\mathrm{d}y. \tag{2.8}$$

Subtracting (2.7) from (2.8) we obtain the DG orthogonality condition

$$\begin{aligned} &\int_{\Gamma_{\text{in}}} v \cdot (\mathbf{F}(U^-) - \mathbf{F}(u)) V \,\mathrm{d}\sigma + \int_{\Gamma_{\text{out}}} v \cdot (\mathbf{F}(U) - \mathbf{F}(u)) V \,\mathrm{d}\sigma \\ &\quad - \int\int_\Delta (\mathbf{F}(U) - \mathbf{F}(u)) \cdot \nabla V \,\mathrm{d}x\,\mathrm{d}y = 0, \quad \forall V \in \mathscr{V}_p. \end{aligned} \tag{2.9}$$

Using the mapping of $\Delta = [0,h]^2$ into the canonical element $\hat{\Delta} = [-1,1]^2$ defined by $x = h(1+\xi)/2$ and $y = h(1+\eta)/2$ and $\hat{u}(\xi,\eta) = u(x(\xi), y(\eta))$ we obtain the DG orthogonality condition (2.9) on the canonical element

$$\begin{aligned} &\int_{\hat{\Gamma}_{\text{in}}} v \cdot (\mathbf{F}(\hat{U}^-) - \mathbf{F}(\hat{u})) \hat{V} \,\mathrm{d}\hat{\sigma} + \int_{\hat{\Gamma}_{\text{out}}} v \cdot (\mathbf{F}(\hat{U}) - \mathbf{F}(\hat{u})) \hat{V} \,\mathrm{d}\hat{\sigma} \\ &\quad - \int\int_{\hat{\Delta}} (\mathbf{F}(\hat{U}) - \mathbf{F}(\hat{u})) \cdot \nabla \hat{V} \,\mathrm{d}\xi\,\mathrm{d}\eta = 0, \quad \forall \hat{V} \in \hat{\mathscr{V}}_p. \end{aligned} \tag{2.10}$$

In the remainder of this paper we omit the $\wedge$ unless we feel it is needed for clarity.

Now, we recall the following two preliminary lemmas.

Lemma 2.1. *If $Q_k \in \mathscr{V}_k$ and $\boldsymbol{\alpha} \in \mathbf{R}^2$ satisfy*

$$\int_{\Gamma_{\text{out}}} \boldsymbol{\alpha} \cdot \boldsymbol{v} Q_k V \,\mathrm{d}\sigma - \int\int_\Delta \boldsymbol{\alpha} \cdot \nabla V Q_k \,\mathrm{d}\xi\,\mathrm{d}\eta = 0, \quad \forall V \in \mathscr{V}_p, \ k \leqslant p, \tag{2.11}$$

then

$$Q_k = 0, \quad k \leqslant p. \tag{2.12}$$

Proof. See Adjerid and Massey [4]. □

Lemma 2.2. *Let $w \in C^\infty(0,h)$ and πw be a p-degree polynomial that interpolates w at Radau points on $[0,h]$. Then the interpolation error*

$$w(x(\xi)) - \pi w(x(\xi)) = \sum_{k=p+1}^{\infty} Q_k^-(\xi)h^k, \tag{2.13a}$$

where

$$Q_{p+1}^-(\xi) = \frac{w^{(p+1)}(0)}{2^{p+1}(p+1)!}(\xi-\xi_0)(\xi-\xi_1)\cdots(\xi-\xi_p) = c_{p+1}R_{p+1}^+(\xi) \tag{2.13b}$$

and

$$Q_k^-(\xi) = R_{p+1}^+(\xi)r_{k-p-1}(\xi), \quad k > p+1, \tag{2.13c}$$

with $r_k(\xi)$ being a polynomial of degree k.

Proof. See Adjerid and Massey [4]. □

Now we are ready to state the main result for nonlinear conservation laws.

Theorem 2.3. *Let u and U be the solution of (1.1) and (2.3), respectively. Then the local finite element error*

$$\epsilon = U - u, \tag{2.14}$$

can be written as

$$\epsilon(\xi,\eta) = \sum_{k=p+1}^{\infty} h^k Q_k(\xi,\eta), \tag{2.15}$$

where

$$Q_{p+1}(\xi,\eta) = \beta_1 R_{p+1}^+(\xi) + \beta_2 R_{p+1}^+(\eta). \tag{2.16}$$

Furthermore, at the outflow boundary of the physical element Δ

$$\int_{\Gamma_{\text{out}}} \boldsymbol{v}\cdot(\mathbf{F}(u) - \mathbf{F}(U))\,\mathrm{d}\sigma = \mathrm{O}(h^{2p+2}). \tag{2.17}$$

If the solution is smooth on $\tilde{\Omega} = \bigcup_{i=1}^{\tilde{N}} \Delta_i$ satisfying (1.5), we have the strong superconvergence

$$\int_{\partial\tilde{\Omega}_{\text{out}}} \boldsymbol{v}\cdot(\mathbf{F}(u) - \mathbf{F}(U))\,\mathrm{d}\sigma = \mathrm{O}(h^{2p+1}). \tag{2.18}$$

Proof. The proof is established using the DG orthogonality condition (2.10). □

First we write the Taylor series of $\mathbf{F}$ about u to obtain

$$\mathbf{F}(U) - \mathbf{F}(u) = \sum_{k=1}^{\infty} \frac{\mathbf{F}^{(k)}(u)}{k!}(U-u)^k. \tag{2.19}$$

Assuming U^- to be the interpolant of u on the inflow boundaries described in (2.4) and using (2.13) we see that on an inflow boundary edge

$$\mathbf{F}(U^-) - \mathbf{F}(u) = \mathbf{F}^{(1)}(u)(U^- - u) + \mathrm{O}(h^{2p+2}). \tag{2.20}$$

The Maclaurin series of $\mathbf{F}^{(1)}(u)$ with respect to h can be written as

$$\mathbf{F}^{(1)}(u) = \sum_{l=0}^{\infty} \mathbf{\Phi}_l^{[1]} h^l, \tag{2.21a}$$

where

$$\mathbf{\Phi}_l^{[1]} = \frac{1}{l!}\frac{\mathrm{d}^l \mathbf{F}^{(1)}(u(x,y))}{\mathrm{d}h^l}\bigg|_{h=0}. \tag{2.21b}$$

Combining (2.13), (2.21) and (2.20) to obtain

$$\boldsymbol{v}\cdot(\mathbf{F}(U^-)-\mathbf{F}(u)) = \begin{cases} h^{p+1}R_{p+1}^+(\xi)\left[\sum_{k=0}^{p} h^k r_{1,k}(\xi)\right] + \mathrm{O}(h^{2p+2}) & \text{on } \Gamma_1, \\ h^{p+1}R_{p+1}^+(\eta)\left[\sum_{k=0}^{p} h^k r_{4,k}(\eta)\right] + \mathrm{O}(h^{2p+2}) & \text{on } \Gamma_4, \end{cases} \tag{2.22}$$

where $r_{1,k}$, $r_{4,k} \in \mathscr{P}_k$.

The Maclaurin series of $U-u$ and $\mathbf{F}^{(k)}(u)$ with respect to h can be written as

$$U-u = \sum_{l=0}^{\infty} Q_l h^l, \tag{2.23a}$$

where

$$Q_l(\xi,\eta) = \frac{1}{l!}\frac{\mathrm{d}^l(U-u)}{\mathrm{d}h^l}\bigg|_{h=0}. \tag{2.23b}$$

We also have

$$\mathbf{F}^{(k)}(u) = \sum_{l=0}^{\infty} \mathbf{\Phi}_l^{[k]} h^l, \tag{2.24a}$$

where

$$\mathbf{\Phi}_l^{[k]}(\xi,\eta) = \frac{1}{l!}\frac{\mathrm{d}^l \mathbf{F}^{(k)}(u)}{\mathrm{d}h^l}\bigg|_{h=0}. \tag{2.24b}$$

Combining (2.19), (2.23) and (2.24) yields

$$\mathbf{F}(U)-\mathbf{F}(u) = \sum_{k=0}^{\infty} \mathbf{W}_k h^k, \tag{2.25}$$

where $\mathbf{W}_k \in \mathscr{P}_k \times \mathscr{P}_k$.

Substituting (2.22) and (2.25) in (2.10) and collecting terms having the same powers of h lead to

$$\begin{aligned}&\sum_{k=0}^{p} h^k\left(\int_{\Gamma_{\text{out}}} \boldsymbol{v}\cdot\mathbf{W}_k V\,\mathrm{d}\sigma - \int\int_{\Delta} \mathbf{W}_k\cdot\nabla V\,\mathrm{d}\xi\,\mathrm{d}\eta\right) \\ &\sum_{k=p+1}^{\infty} h^k\left(\int_{\Gamma_{\text{in}}} \boldsymbol{v}\cdot\mathbf{W}_k^- V\,\mathrm{d}\sigma + \int_{\Gamma_{\text{out}}} \boldsymbol{v}\cdot\mathbf{W}_k V\,\mathrm{d}\sigma - \int\int_{\Delta} \mathbf{W}_k\cdot\nabla V\,\mathrm{d}\xi\,\mathrm{d}\eta\right) = 0, \quad \forall V\in\mathscr{V}_p,\end{aligned} \tag{2.26}$$

where using (2.22) we have

$$\boldsymbol{v}\cdot\mathbf{W}_k^- = \begin{cases} R_{p+1}^+(\xi)r_{1,k-p-1}(\xi) & \text{on } \Gamma_1, \\ R_{p+1}^+(\eta)r_{4,k-p-1}(\eta) & \text{on } \Gamma_4, \end{cases} \quad p+1 \leqslant k \leqslant 2p+1. \tag{2.27}$$

The O(1) term in (2.26) with $V = 1$ yields

$$\mathbf{W}_0 = Q_0\left(\sum_{l=0}^{\infty}\frac{\mathbf{\Phi}_0^{[l+1]}Q_0^l}{(l+1)!}\right) = 0, \tag{2.28}$$

which in turn leads to $Q_0 = 0$. We note that, for instance, for $\mathbf{F}(u) = [u^2/2, u]^t$ there exists $Q_0 \neq 0$ solution of (2.28) which corresponds to a *nonphysical* DG solution with $\epsilon = \mathrm{O}(1)$. Here, we will not consider such *non-physical* solutions.

By induction the $\mathrm{O}(h^k)$, $k < p+1$ leads to

$$\int_{\Gamma_{\text{out}}} \boldsymbol{v}\cdot\mathbf{\Phi}_0^{[1]}Q_k V\,\mathrm{d}\sigma - \int\int_{\Delta} Q_k\mathbf{\Phi}_0^{[1]}\cdot\nabla V\,\mathrm{d}\xi\,\mathrm{d}\eta = 0, \quad \forall V\in\mathscr{V}_p. \tag{2.29}$$

Applying Lemma 2.1 we establish that $Q_k = 0$, $k = 1, 2, \ldots, p$.

Following the same line of reasoning as in [4], we show that the leading term Q_{p+1} can be split as

$$Q_{p+1} = \frac{1}{(p+1)!}\frac{\mathrm{d}^{p+1}(U-u)}{\mathrm{d}h^{p+1}}(\xi,\eta)\bigg|_{h=0} = \check{Q}_{p+1} + \tilde{Q}_p, \tag{2.30a}$$

where $\tilde{Q}_p(\xi,\eta)\in\mathscr{V}_p$ and

$$\check{Q}_{p+1} = c_{p+1}\frac{1}{2^{p+1}(p+1)!}\frac{\partial^{p+1}u}{\partial x^{p+1}}(0,0)R_{p+1}^{+}(\xi) + c_{p+1}\frac{1}{2^{p+1}(p+1)!}\frac{\partial^{p+1}u}{\partial y^{p+1}}(0,0)R_{p+1}^{+}(\eta). \tag{2.30b}$$

Substituting (2.30) in the $\mathrm{O}(h^{p+1})$ term of the series (2.26) leads to

$$\begin{aligned}&\int_{\Gamma_{\text{in}}}\mathbf{\Phi}_0^{[1]}\cdot\boldsymbol{v}Q_{p+1}^{-}V\,\mathrm{d}\sigma + \int_{\Gamma_{\text{out}}}\mathbf{\Phi}_0^{[1]}\cdot\boldsymbol{v}\check{Q}_{p+1}V\,\mathrm{d}\sigma - \int\int_{\Delta}\mathbf{\Phi}_0^{[1]}\cdot\nabla V\check{Q}_{p+1}\,\mathrm{d}\xi\,\mathrm{d}\eta + \int_{\Gamma_{\text{out}}}\mathbf{\Phi}_0^{[1]}\cdot\boldsymbol{v}\tilde{Q}_pV\,\mathrm{d}\sigma\\ &\quad - \int\int_{\Delta}\mathbf{\Phi}_0^{[1]}\cdot\nabla V\tilde{Q}_p\,\mathrm{d}\xi\,\mathrm{d}\eta = 0, \quad \forall V\in\mathscr{V}_p.\end{aligned} \tag{2.31}$$

Using (2.13) and (2.30b) we can show that

$$\int_{\Gamma_{\text{in}}}\mathbf{\Phi}_0^{[1]}\cdot\boldsymbol{v}Q_{p+1}^{-}V\,\mathrm{d}\sigma + \int_{\Gamma_{\text{out}}}\mathbf{\Phi}_0^{[1]}\cdot\boldsymbol{v}\check{Q}_{p+1}V\,\mathrm{d}\sigma - \int\int_{\Delta}\mathbf{\Phi}_0^{[1]}\cdot\nabla V\check{Q}_{p+1}\,\mathrm{d}\xi\,\mathrm{d}\eta = 0, \quad \forall V\in\mathscr{V}_p. \tag{2.32}$$

Combining (2.31) and (2.32) with Lemma 2.1 leads to $\tilde{Q}_p = 0$. Using (2. 30) we establish (2.16).

Using (2.13) we can show that

$$\int_{\Gamma_{\text{in}}}\mathbf{\Phi}_0^{[1]}\cdot\boldsymbol{v}Q_k^{-}V\,\mathrm{d}\sigma = 0, \quad \forall V\in\mathscr{V}_{2p-k}, \quad k = p+1,\ldots,2p. \tag{2.33}$$

Using (2.33), the $\mathrm{O}(h^k)$, $p+1\leqslant k\leqslant 2p$, term of (2.26) yields

$$\int_{\Gamma_{\text{out}}}\mathbf{\Phi}_0^{[1]}\cdot\boldsymbol{v}Q_kV\,\mathrm{d}\sigma - \int\int_{\Delta}\mathbf{\Phi}_0^{[1]}\cdot\nabla VQ_k\,\mathrm{d}\xi\,\mathrm{d}\eta = 0, \quad \forall V\in\mathscr{V}_{2p-k}. \tag{2.34}$$

Testing against $V = 1$ we obtain

$$\int_{\Gamma_{\text{out}}}\mathbf{\Phi}_0^{[1]}\cdot\boldsymbol{v}Q_k\,\mathrm{d}\sigma = 0, \quad k = p+1,\ldots,2p, \tag{2.35}$$

which establishes (2.17).

Next we prove global superconvergence by showing that on every element Δ

$$\int_{\Gamma_{\text{in}}} \boldsymbol{v} \cdot (\mathbf{F}(U^-) - \mathbf{F}(u))\,\mathrm{d}\sigma + \int_{\Gamma_{\text{out}}} \boldsymbol{v} \cdot (\mathbf{F}(U) - \mathbf{F}(u))\,\mathrm{d}\sigma = 0. \tag{2.36}$$

Summing over all elements $\Delta_i \subset \tilde{\Omega}$ we obtain

$$\int_{\partial\tilde{\Omega}_{\text{in}}} \boldsymbol{v} \cdot (\mathbf{F}(U^-) - \mathbf{F}(u))\,\mathrm{d}\sigma + \int_{\partial\tilde{\Omega}_{\text{out}}} \boldsymbol{v} \cdot (\mathbf{F}(U) - \mathbf{F}(u))\,\mathrm{d}\sigma = 0. \tag{2.37}$$

Using (2.13) leads to (2.18). $\square$

Next, we will describe similar results for problems of the form (1.2) where the DG weak formulation consists of determining $U(x,y) \in \mathscr{V}_p$ on Δ such that

$$\int_{\Gamma_{\text{in}}} \boldsymbol{v} \cdot (\mathbf{F}(U^-) - \mathbf{F}(U))V\,\mathrm{d}\sigma + \int\int_{\Delta} [\nabla \cdot \mathbf{F}(U) + \phi(U) - h(x,y)]V\,\mathrm{d}x\,\mathrm{d}y = 0, \quad \forall V \in \mathscr{V}_p. \tag{2.38}$$

In the following theorem we state a superconvergence result for nonlinear hyperbolic problem with reaction terms.

Theorem 2.4. *Let u and U be the solution of (1.2) and (2.38), respectively. If u, $\mathbf{F}$ and ϕ are analytic functions, then the local error estimates (2.15) and (2.16) hold. Furthermore, we have the following superconvergence results on the first inflow element*

$$\int_{\Gamma_{\text{out}}} \boldsymbol{v} \cdot (\mathbf{F}(U) - \mathbf{F}(u)) = \mathrm{O}(h^{\min(2p+2,p+4)}) \tag{2.39}$$

and

$$\int_{\Gamma_{\text{out}}} \boldsymbol{v} \cdot (\mathbf{F}(U) - \mathbf{F}(u))\,\mathrm{d}\sigma + \int\int_{\Delta} [\phi(U) - \phi(u)]\,\mathrm{d}x\,\mathrm{d}y = \mathrm{O}(h^{2p+2}). \tag{2.40}$$

If the solution is smooth on $\tilde{\Omega}$ satisfying (1.5) and $\tilde{\Omega} = \bigcup_{i=1}^{\tilde{N}} \Delta_i$, we have the strong superconvergence

$$\int_{\partial\tilde{\Omega}_{\text{out}}} \boldsymbol{v} \cdot (\mathbf{F}(U) - \mathbf{F}(u))\,\mathrm{d}\sigma + \int\int_{\tilde{\Omega}} [\phi(U) - \phi(u)]\,\mathrm{d}x\,\mathrm{d}y = \mathrm{O}(h^{2p+1}). \tag{2.41}$$

Proof. The DG orthogonality condition is

$$\int_{\Gamma_{\text{in}}} \boldsymbol{v} \cdot (\mathbf{F}(U^-) - \mathbf{F}(u))V\,\mathrm{d}\sigma + \int_{\Gamma_{\text{out}}} \boldsymbol{v} \cdot (\mathbf{F}(U) - \mathbf{F}(u))V\,\mathrm{d}\sigma \\ - \int\int_{\Delta} (\mathbf{F}(U) - \mathbf{F}(u)) \cdot \nabla V + [\phi(u) - \phi(U)]V\,\mathrm{d}x\,\mathrm{d}y = 0, \quad \forall V \in \mathscr{V}_p. \tag{2.42}$$

On the canonical element $[-1,1]^2$ (2.42) becomes

$$\int_{\Gamma_{\text{in}}} \boldsymbol{v} \cdot (\mathbf{F}(U^-) - \mathbf{F}(u))V\,\mathrm{d}\sigma + \int_{\Gamma_{\text{out}}} \boldsymbol{v} \cdot (\mathbf{F}(U) - \mathbf{F}(u))V\,\mathrm{d}\sigma \\ - \int\int_{\Delta} (\mathbf{F}(U) - \mathbf{F}(u)) \cdot \nabla V + \frac{h}{2}[\phi(u) - \phi(U)]V\,\mathrm{d}\xi\,\mathrm{d}\eta = 0, \quad \forall V \in \mathscr{V}_p. \tag{2.43}$$

Applying Taylor series to ϕ about u we have

$$\phi(u) - \phi(U) = -a(u)\epsilon - \frac{\epsilon^2}{2}\phi''(\bar{u}), \quad a(u) = \phi'(u). \tag{2.44a}$$

The Maclaurin series of $a(u)$ with respect to h yields

$$a(u) = 2\sum_{k=0}^{\infty} h^k \overline{Q}_k(\xi,\eta), \quad \overline{Q}_k(\xi,\eta) = \frac{1}{2}\frac{\phi^{(k+1)}(u(x(\xi),y(\eta)))}{k!}\frac{\mathrm{d}^k u}{\mathrm{d}h^k}(x(\xi),y(\eta))\bigg|_{h=0} \in \mathscr{P}_k. \tag{2.44b}$$

Now we substitute (2.44), (2.22) and (2.25) in (2.43), collect terms having the same powers of h to obtain

$$\begin{aligned}&\left(\int_{\Gamma_{\text{out}}} \boldsymbol{v}\cdot\mathbf{W}_0 V\,\mathrm{d}\sigma - \int\int_{\Delta} \mathbf{W}_0\cdot\nabla V\,\mathrm{d}\xi\,\mathrm{d}\eta\right)\\ &\quad + \sum_{k=1}^{p} h^k\left(\int_{\Gamma_{\text{out}}} \boldsymbol{v}\cdot\mathbf{W}_k V\,\mathrm{d}\sigma - \int\int_{\Delta}[\mathbf{W}_k\cdot\nabla V - Z_{k-1}V]\,\mathrm{d}\xi\,\mathrm{d}\eta\right)\\ &\quad + \sum_{k=p+1}^{\infty} h^k\left(\int_{\Gamma_{\text{in}}} \boldsymbol{v}\cdot\mathbf{W}_k^- V\,\mathrm{d}\sigma + \int_{\Gamma_{\text{out}}} \boldsymbol{v}\cdot\mathbf{W}_k V\,\mathrm{d}\sigma - \int\int_{\Delta}[\mathbf{W}_k\cdot\nabla V - Z_{k-1}V]\,\mathrm{d}\xi\,\mathrm{d}\eta\right) = 0, \quad \forall V \in \mathscr{V}_p,\end{aligned} \tag{2.45}$$

where

$$Z_k = \sum_{l=0}^{k} \overline{Q}_l Q_{k-l} \tag{2.46}$$

and $\boldsymbol{v}\cdot\mathbf{W}_k^-$ is given in (2.27).

Following the same line of reasoning as in Theorem 2.3 we prove (2.15) and (2.16) for problems with a nonlinear reaction term. We note that the term in (2.44a) involving ϵ^2 is higher order and does not contribute to our leading terms and that

$$Z_m = \begin{cases} 0 & \text{if } m \leqslant p,\\ \sum_{l=0}^{m-p-1} \overline{Q}_l Q_{m-l}, & \text{if } m \geqslant p+1. \end{cases} \tag{2.47}$$

We prove the strong superconvergence result (2.39) for nonlinear hyperbolic problems with reaction terms by setting $V = 1$ in (2.45). Using (2.13), (2.15) and (2.16), to obtain

$$\int_{\Gamma_{\text{out}}} \boldsymbol{v}\cdot\mathbf{\Phi}_0^{[1]} Q_{p+1}\,\mathrm{d}\sigma = 0. \tag{2.48}$$

Setting $V = 1$ in the $\mathrm{O}(h^k)$, $k > p+1$ in (2.45) leads to

$$\int_{\Gamma_{\text{in}}} \boldsymbol{v}\cdot\mathbf{\Phi}_0^{[1]} Q_k^-\,\mathrm{d}\sigma + \int_{\Gamma_{\text{out}}} \boldsymbol{v}\cdot\mathbf{\Phi}_0^{[1]}\,\mathrm{d}\sigma + \int\int_{\Delta} Z_{k-1}\,\mathrm{d}\xi\,\mathrm{d}\eta = 0. \tag{2.49}$$

Using (2.22) and (2.47) the $\mathrm{O}(h^{p+2})$ term leads to

$$\int_{\Gamma_{\text{out}}} \boldsymbol{v}\cdot\mathbf{\Phi}_0^{[1]} Q_{p+2}\,\mathrm{d}\sigma = 0. \tag{2.50}$$

The $\mathrm{O}(h^{p+3})$ term yields

$$\int_{\Gamma_{\text{out}}} \boldsymbol{v}\cdot\mathbf{\Phi}_0^{[1]} Q_{p+3}\,\mathrm{d}\sigma = -\int\int_{\Delta} \overline{Q}_0 Q_{p+2}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{2.51}$$

which is not necessarily zero. Thus, we establish (2.39).

Now, using (2.42) with $V = 1$ and (2.22) establishes (2.40).

Letting $V=1$ in (2.42) and summing over all elements $\Delta_i \subset \tilde{\Omega}$ lead to

$$\int_{\partial\tilde{\Omega}_{\text{in}}} \boldsymbol{v}\cdot(\mathbf{F}(U^-)-\mathbf{F}(u))\,\mathrm{d}\sigma + \int_{\partial\tilde{\Omega}_{\text{out}}} \boldsymbol{v}\cdot(\mathbf{F}(U)-\mathbf{F}(u))\,\mathrm{d}\sigma + \int\int_{\tilde{\Omega}}[\phi(U)-\phi(u)]\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{2.52}$$

Applying (2.13) and (2.22) yields (2.41) which completes the proof of Theorem 2.4. $\square$

3. Numerical examples

We will consider three examples to validate the superconvergence results of Section 2 for smooth and discontinuous solutions.

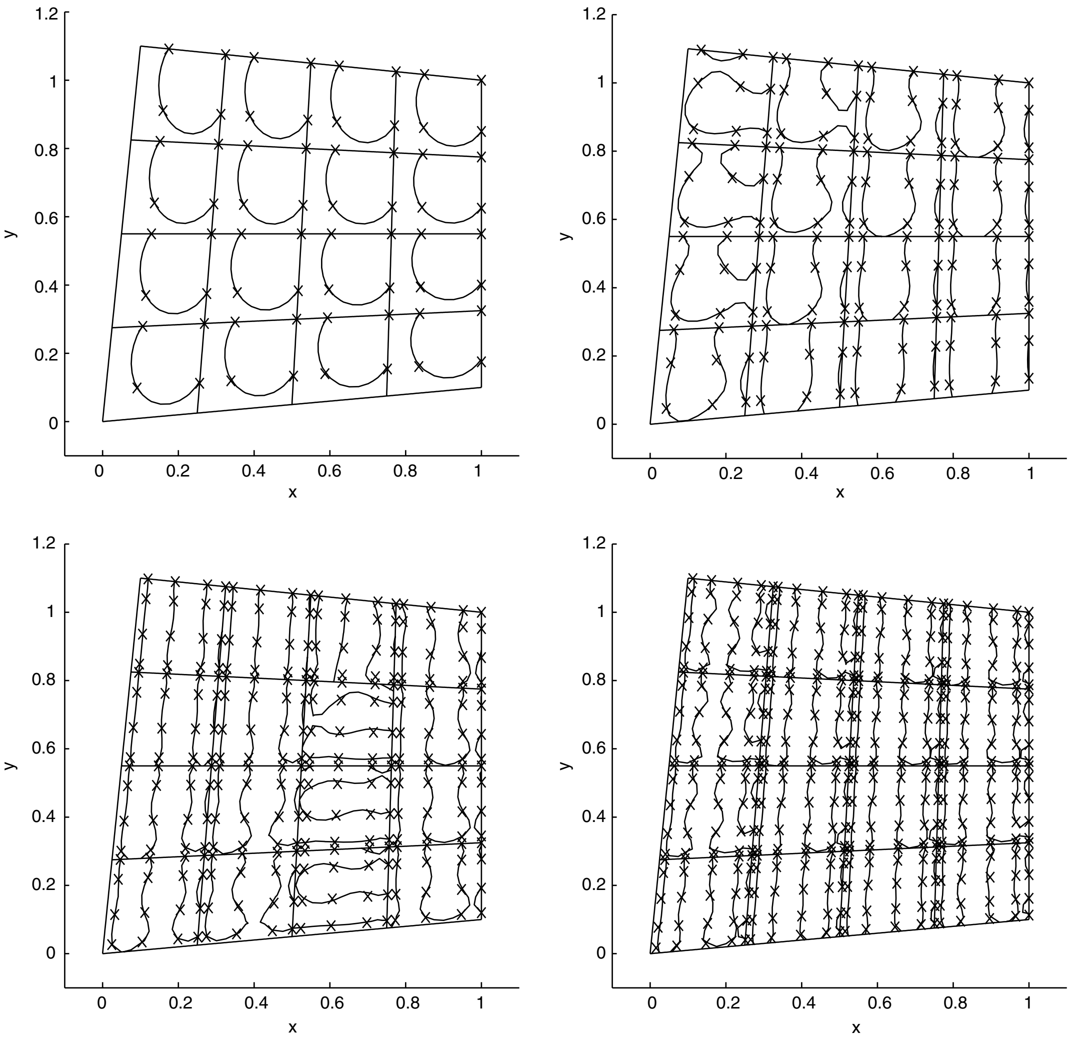

Fig. 1. Zero-level curves of the error for Example 1 using $p = 1,2,3,4$ (upper left to lower right). Radau points are shown with an '×'.

Example 1. We consider the nonlinear Burger's equation

$$u_y + uu_x = f(x,y), \quad (x,y) \in \Omega, \tag{3.1a}$$

where Ω is the quadrilateral $P_1P_2P_3P_4$ where $P_1 = (0,0)$, $P_2 = (1,1)$, $P_3 = (2,1)$ and $P_4 = (-0.5,2)$. We select the boundary conditions and f such that the exact solution is

$$u(x,y) = \sqrt{3 + 2x^2 + y^2}. \tag{3.1b}$$

We solve this problem on a uniform mesh having 16 elements with $p = 1,2,3,4$ and plot the 0-level curves of the discontinuous Galerkin error in Fig. 1 with Radau points marked by x.

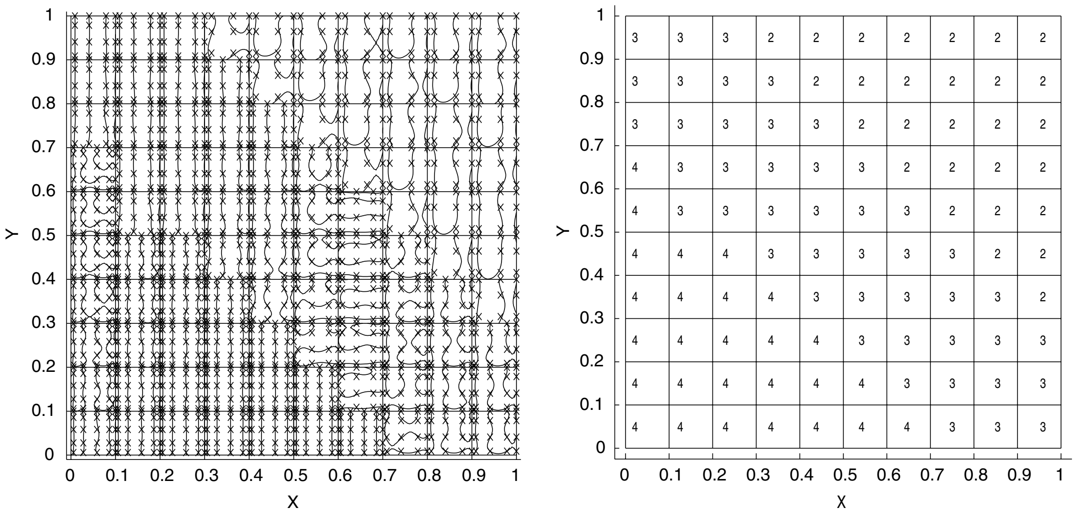

Fig. 2. Zero-level curves of the error for Example 1 using nonuniform polynomial degree with Radau points shown with an '×' (left). Distribution of the polynomial degree of the finite element solution (right).

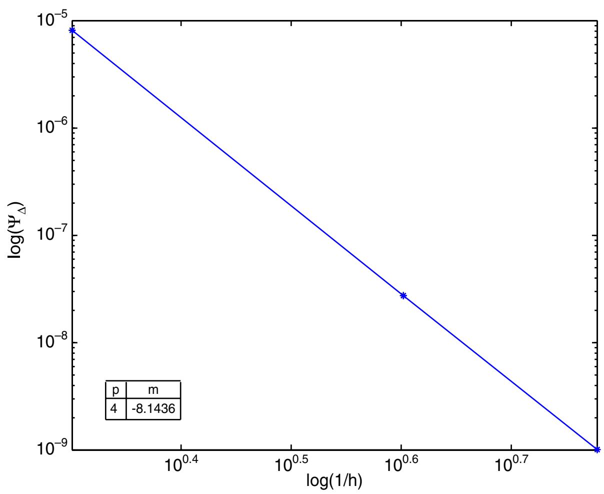

Fig. 3. The error in the flux Ψ_Δ on the outflow boundary of the first element versus $1/h$ in the log–log scale for Example 2.

These results show that the solution is superconvergent at Radau points which is in full agreement with Theorem 2.3. Next we solve (3.1) with nonuniform p as shown in Fig. 2. The results shown in Fig. 2 indicate that the superconvergence results of Section 2 are still valid for nonuniform polynomial degree starting with higher degree on elements at the inflow boundary of the domain and lower polynomial degree on elements at the outflow boundary. These results are yet to be proved for nonuniform polynomial degree.

Example 2. In order to show that the superconvergence result (2.39) is optimal, we consider the linear hyperbolic problem with a reaction term

$$u_x + 2u_y + u = f(x,y), \quad (x,y) \in [0,1]^2, \tag{3.2a}$$

where the boundary conditions and f are selected such that the exact solution is

$$u(x,y) = (1 + x + y)^7. \tag{3.2b}$$

We solve (3.2) on uniform meshes having 4, 16 and 36 square elements with $p = 4$ and plot

$$\Psi_\Delta = \left| \int_{\Gamma_{\text{out}}} [1,2] \cdot \boldsymbol{v}(u - U)\,\mathrm{d}\sigma \right|$$

versus $1/h$ in Fig. 3. As predicted by Theorem 2.4, these results show an $O(h^{\min(2p+2,p+4)})$ superconvergence rate of the flux on the outflow boundary of the first inflow element.

Example 3. Let us consider the homogeneous inviscid Burger's equation

$$u_y + uu_x = 0, \quad (x,y) \in [-1,1] \times [0,1.999], \tag{3.3a}$$

subject to the initial condition

$$g_0(x,0) = 1 + \sin(\pi x)/2. \tag{3.3b}$$

We select $g_1(0,y)$ such that the unique entropy solution is periodic and forms a shock discontinuity at the point $(\frac{2}{\pi} - 1, \frac{2}{\pi})$ which propagates along the line $y = x + 1$. We solve this problem on meshes having 5×5, 14×10, 21×15, 28×20, 35×25, 42×30, and 140×100 elements with $p = 1,2$. We plot the zero-level curves for the error in Fig. 5. We compute the maximum errors at Radau points in five different regions

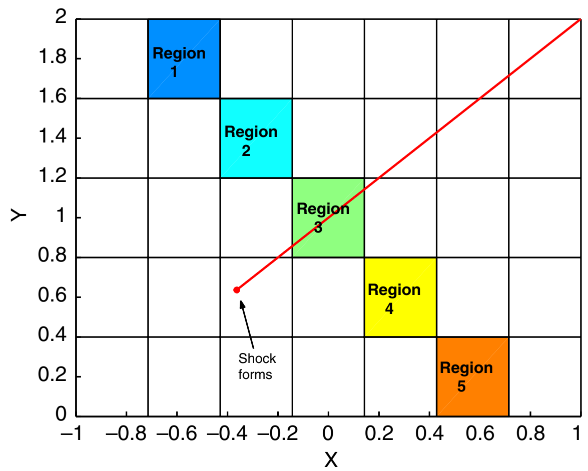

Fig. 4. Regions 1–5 for problem (3.3).

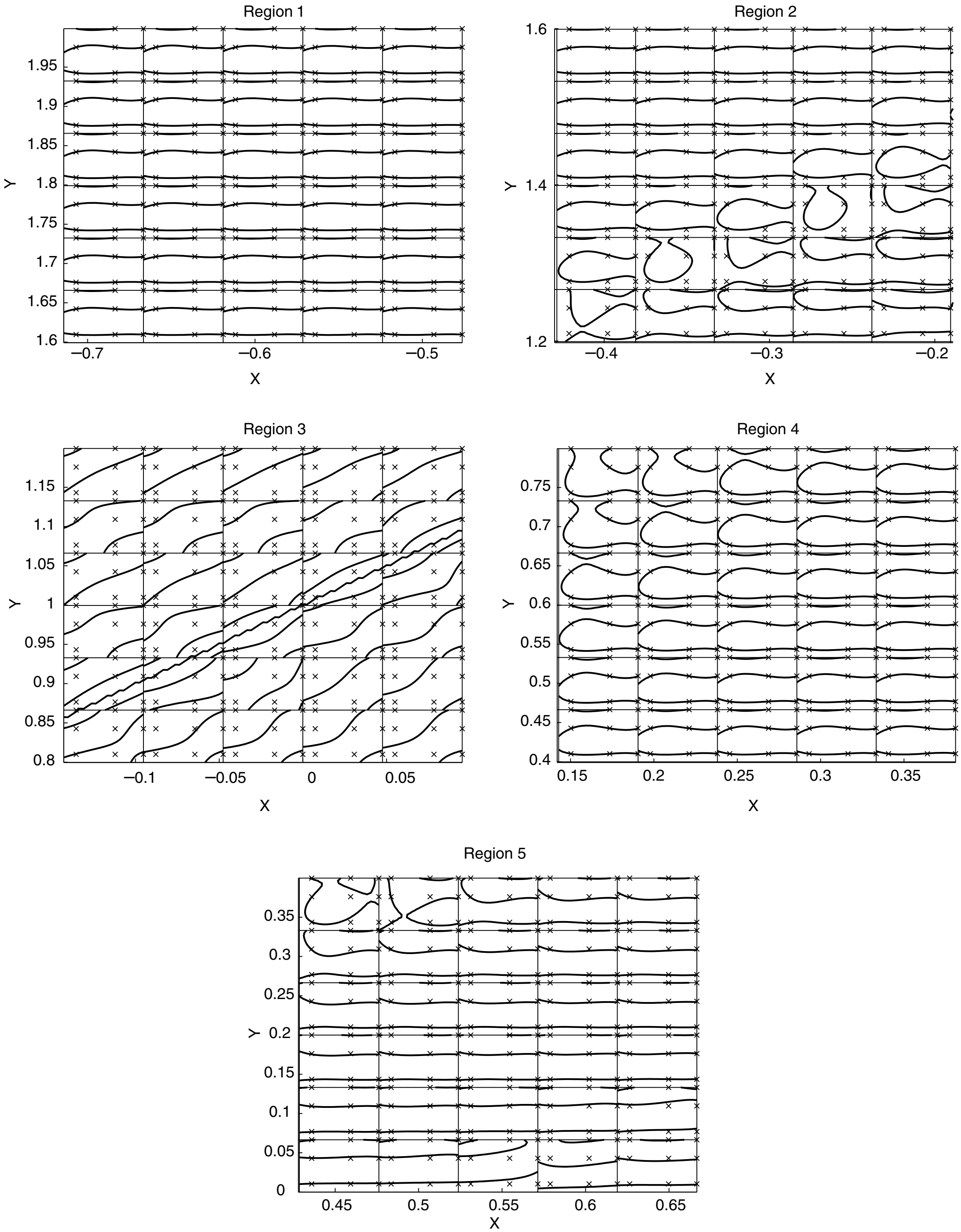

Fig. 5. Zero-level curves of the error for problem (3.3) on 42×30 mesh and $p = 2$ on Regions 1–5 (upper left to lower center). Radau points are shown with an '×'.

shown in Fig. 4 and present the results for all meshes and orders in Table 1. We also plot the maximum errors versus mesh size for $p = 1,2$ in Fig. 6. We note that the convergence rates are close to the optimal $O(h^{p+2})$ superconvergence rates in Regions 1, 2 , 4 and 5, for $p = 1$ and $p = 2$, while there is no convergence in Region 3 which contains the shock discontinuity. These results are in agreement with the superconvergence results of Theorem (2.3) for the entropy solution in regions away from the shock discontinuity. We note that high-order, $p > 0$, DG solutions are known to develop spurious oscillations near shock discontinuities which explains the nonconvergence in Region 3. Spurious oscillations may be eliminated using, for

Table 1
Maximum errors at Radau points for problem (3.3) in Regions 1–5

$N \times M$	Region 1	Region 2	Region 3	Region 4	Region 5
$p = 1$					
7×5	2.3787e−03	6.1577e−02	4.4390e−01	3.0670e−02	4.7646e−02
14×10	5.7487e−05	3.8567e−03	4.4521e−01	2.7064e−03	9.8580e−03
21×15	1.7168e−05	1.6498e−03	4.3000e−01	6.9151e−04	4.4972e−03
28×20	7.4312e−06	1.0519e−03	5.8166e−01	3.1665e−04	2.0990e−03
35×25	3.8739e−06	1.2146e−04	5.3141e−01	1.7085e−04	1.1055e−03
42×30	2.2684e−06	1.1820e−04	5.8720e−01	1.0250e−04	7.2652e−04
140×100	6.3664e−08	4.0439e−06	6.0357e−01	3.2124e−06	2.1123e−05
Rate	2.9636	2.4543	−0.091848	2.8665	2.8549
$p = 2$					
7×5	1.0780e−03	3.5124e−02	4.5072e−01	2.3503e−02	5.4503e−03
14×10	2.1984e−06	9.6276e−03	4.5917e−01	1.1295e−03	1.5127e−03
21×15	2.0416e−07	2.0612e−03	6.4203e−01	4.2546e−04	4.2145e−04
28×20	6.2627e−08	1.9124e−04	4.8639e−01	7.7144e−05	1.2900e−04
35×25	2.5654e−08	4.4611e−05	6.2678e−01	4.8348e−06	5.2180e−05
42×30	1.2284e−08	2.1228e−05	6.1557e−01	6.0801e−07	2.8264e−05
140×100	9.7912e−11	8.1746e−09	6.7466e−01	5.5857e−09	2.4411e−07
Rate	4.0167	6.2070	−0.053101	4.8788	3.8699

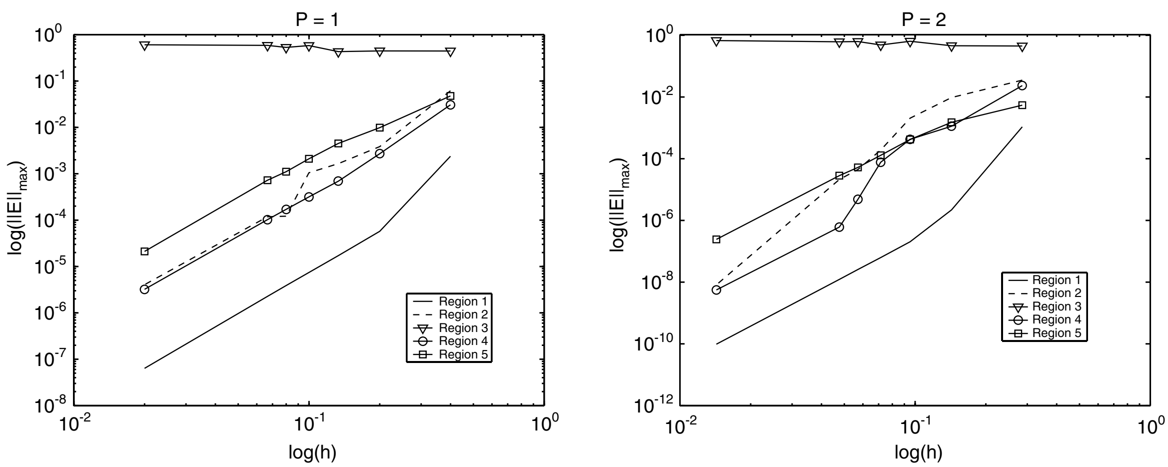

Fig. 6. Rates of convergence for problem (3.3) on meshes have 7×5, 14×10, 21×15, 28×20, 35×25, 42×30 and 140×100 elements on Regions 1–5 with $p = 1$ (left) and $p = 2$ (right).

instance, shock capturing [21] or limiting [10,11] techniques. Adjerid et al. [1] used the limiting technique of Biswas et al. [10] for one-dimensional nonlinear problems that eliminates spurious oscillations near discontinuities while preserving superconvergence in smooth regions. We note that reliable shock detection is key to successful adaptive limiting strategies [23].

4. Conclusions

We extended the results of Adjerid and Massey [4] to nonlinear conservation laws. We proved that the discontinuous Galerkin finite element solution is $O(h^{p+2})$ superconvergent at the Radau points. We also showed that locally the flux is $O(h^{2p+2})$ superconvergent on average on the outflow boundary of the first inflow element. In the presence of reaction terms we proved similar superconvergence results for the solution at Radau points and an $O(h^{\min(2p+2,p+4)})$ superconvergence rate for the flux on average at the outflow boundary of the first inflow element. Furthermore, we showed that on sub-domains satisfying (1.5) the sum of the flux on the outflow boundary and the reaction term is $O(h^{2p+1})$ superconvergent. The strong superconvergence of the flux yields superconvergence of the solution at Radau points on every element. As shown in Adjerid and Massey [4], these superconvergence results for discontinuous finite element solutions may be used to construct asymptotically correct a posteriori error estimates for steering adaptive finite element methods. Numerical computations of Adjerid and Klauser [3] suggest that similar superconvergence results still hold for local discontinuous Galerkin solutions of convection–diffusion problems. The error analysis described in this manuscript has been extended to semi-discrete DG methods for transient nonlinear scalar hyperbolic conservation laws [2]. A more difficult problem is to extend the analysis to arbitrary elements in smooth-solution regions and to the shock region with limiting or shock capturing. We will investigate the superconvergence properties of the shock capturing discontinuous Galerkin method [21,12] for hyperbolic systems. The preliminary work of Krivodonova and Flaherty [22] shows superconvergence of the flux on element outflow boundaries for general unstructured triangular meshes, however, no pointwise superconvergence has been observed.

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