Convergence of Laplacian diffusion versus resolution of an ocean model

Alan J. Wallcraft, A. Birol Kara, and Harley E. Hurlburt

Oceanography Division, Naval Research Laboratory, Stennis Space Center, Mississippi, USA

Received 21 January 2005; revised 18 February 2005; accepted 8 March 2005; published 7 April 2005.

[1] This paper presents a convergence study for second order finite difference Laplacian diffusion used in ocean models. For demonstration, ocean model simulations are performed over a rectangular domain, based on the North Pacific subtropical gyre region with grid resolution between 1/2° and 1/32° and with horizontal eddy viscosity coefficient (A_H) ranging from 8000 to 30 m^2 s^{-1}. A range of A_H which is appropriate for useful model simulations of an oceanic domain is found to exist. This range is determined by examining the spatial patterns of Eddy kinetic energy and mean sea surface height. The results fall into three broad categories: (a) converged, (b) converging, and (c) numerical problems. Solutions in the "converged" category do not change with increased grid resolution, and solutions in the "numerical problems" category exhibit distinct differences to the converged result at the same A_H. **Citation:** Wallcraft, A. J., A. B. Kara, and H. E. Hurlburt (2005), Convergence of Laplacian diffusion versus resolution of an ocean model, *Geophys. Res. Lett.*, *32*, L07604, doi:10.1029/2005GL022514.

1. Introduction

[2] In all numerical ocean models it is necessary to parameterize the effects of unresolved scales of motion. Since the horizontal (isopycnal) and vertical scales are so different they are usually treated separately, and some form of Laplacian diffusion is a common choice for the horizontal parameterization [e.g., *Berloff and McWilliams*, 2002]. Diffusion on a fixed Eulerian grid is complicated by issues concerning the difference between "horizontal" in grid coordinates and in isopycnal coordinates [e.g., *Killworth*, 1997].

[3] Layered models are already using isopycnal, or approximately isopycnal, coordinates and hence give a cleaner separation between horizontal and vertical diffusion [*Smith and Gent*, 2004]. However, Lagrangian layers provide several possible forms for the Laplacian diffusion term. In fact, there is only one correct form which is expressible as divergence of the viscous stress tensor [e.g., *Shchepetkin and O'Brien*, 1996]. In Cartesian coordinates, the form of Laplacian diffusion used here is $A_H \nabla^2 (h \vec{V})$, and the correct form is $A_H (\nabla \cdot (h\nabla)) \vec{V}$, where A_H is the horizontal eddy viscosity coefficient (m^2 s^{-1}), $\vec{V}$ is the layer velocity (m s^{-1}) and h is layer thickness (m). These two forms are equivalent if h is constant as would be the case in z-level models [e.g., *Griffies et al.*, 2000].

[4] In this paper, a convergence study of the second order finite difference horizontal eddy viscosity variability is undertaken in the idealized Pacific Ocean by examining simulations from an ocean model. In numerical ocean modeling studies, "convergence" can have at least two meanings. Here, we have taken it to mean convergence to the continuum limit of the finite difference equations as the horizontal grid resolution is refined. An alternative meaning, of more practical interest to ocean modelers, is convergence on the actual behavior of the ocean region under study. In this stronger sense, convergence implies no significant difference in fidelity to the real ocean between the best choice of model parameters at one resolution and the best choice of model parameters at a finer resolution. Here, "best choice" can obviously include a lower eddy viscosity on the finer grid but it can also include a more accurate coastline and bottom topography.

2. Ocean Model Application

[5] The numerical model, Naval Research Laboratory (NRL) Layered Ocean Model (NLOM), uses a primitive equation layered formulation where the equations have been vertically integrated through each Lagrangian layer. Prognostic variables are layer density, layer thickness, and layer volume transport per unit width (layer velocity times layer thickness) as described by *Wallcraft et al.* [2003]. Our experience has been that horizontal diffusion is largely independent of the number of layers used. Two isopycnal layers are typically sufficient at an appropriate horizontal resolution and value of A_H to yield relevant (semi-quantitatively) representations of mesoscale eddies and western boundary currents in subtropical-gyre-like model domains [*Schmitz and Thompson*, 1993]. The simplicity of a layered model formulation permits many numerical experiments to be performed at comparatively high horizontal resolutions [*Hurlburt and Hogan*, 2000] with a variety of values of A_H, such that western boundary current path and structures, including recirculating gyres are also realistically represented. Thus, subtropical gyre simulations in this study use a two layer model, a configuration which has the additional advantage that mixing between layers (an additional source of damping) is minimized.

[6] Since we are concerned with convergence to the continuum limit of the ocean model equations in this study, the model domain need not be an actual ocean basin. However, the results must be representative of actual ocean basins since these are our primary interest. The model simulations are performed using a rectangular-shaped area of the Pacific Ocean, corresponding to the subtropical gyre region, with closed boundaries. It is based on a similar subtropical gyre region, that included a realistic representation of the Japan/East Sea. For this study, the Japan/East Sea has been converted into a plateau. Removing the Japan/East Sea is necessary because the straits are not resolvable on all

Table 1. Grid Resolutions Used in the Model Simulations of a Rectangular Pacific Region[a]

Resolution	Δx (deg)	Δy (deg)	Δx (km)	Δy (km)
1/2°	0.703125	0.5	55.3	55.6
1/4°	0.351569	0.25	27.6	27.8
1/8°	0.175781	0.125	13.8	13.9
1/16°	0.087890	0.0625	6.9	6.9
1/32°	0.043945	0.03125	3.5	3.5

the grids used here. Rather than converting Japan to a seamount, it would have been possible to convert the Japan/East Sea to "land". However, realistic simulations of the Pacific subtropical gyre are not possible without the Japan/East Sea [*Hogan and Hurlburt*, 2000]. Rectangular regions are less expensive to run, and more importantly a rectangular region allows prototyping of new numerical schemes without requiring implementation of general coastline boundary conditions. The disadvantage of a rectangular region is that it may increase the tendency for western boundary currents to "overshoot" the expected separation point from the boundary. However, overshoot is also common in nonrectangular regions.

[7] Here, all simulations use a layered Laplacian diffusion coefficient, which was standard in earlier versions of the model [e.g., *Hurlburt et al.*, 1996], while more recent NLOM simulations of actual ocean basins use stress tensor Laplacian diffusion coefficient [e.g., *Kara et al.*, 2003; *Wallcraft et al.*, 2003; *Kara et al.*, 2004]. In practice, little difference was found between the various Laplacian formulations. In particular, repeating a few simulations from this study with the exact form of Laplacian diffusion showed no change in the model simulations. A layered Laplacian diffusion is used because it is significantly less computationally expensive.

[8] The model simulations are performed using grid resolutions of 1/2°, 1/4°, 1/8°, 1/16° and 1/32°. Corresponding grid spacings in km are given in Table 1. Thirty two simulations are spun up to statistical equilibrium with coefficients of horizontal eddy viscosity, A_H, between 8000 and 30 $m^2\ s^{-1}$. Each two layer simulation uses identical realistic bottom topography from a modified version of the Earth Topography Five Minute Grid (ETOPO5) [*National Oceanic and Atmospheric Administration*, 1988] and the wind stress forcing is *Hellerman and Rosenstein* [1983].

3. Convergence of Eddy Viscosity

[9] In the model analysis (see section 4), the model simulations are subjectively divided into three broad categories: (a) converged, (b) converging, and (c) numerical problems. Since simulations are nondeterministic due to flow instabilities, only statistical convergence is sought (i.e., mean and variability). Solutions in the "converged" category do not change statistically with increased grid resolution. They are essentially identical statistically to very high resolution at the same eddy viscosity. Solutions in the "numerical problems" category exhibit distinct differences to the converged result at the same A_H. Solutions in the "converging" category are similar to the converged result, but with quantitative differences. It is possible for the converged solution to change drastically as the eddy viscosity is reduced. This can happen when transitioning from non-eddy-resolving to eddy-resolving eddy viscosities. It can also happen at lower eddy viscosities as new energy pathways become dominant.

[10] Figure 1 summarizes the results on a graph of Laplacian diffusion values with respect to grid resolution. No eddies are present above about A_H = 1000 $m^2\ s^{-1}$, and eddies are always present below about A_H = 500 $m^2\ s^{-1}$. Since the categorization is subjective, the boundaries between converged and converging and between converging and numerical problems are not sharply defined but they appear to be proportional to the square of the grid resolution. The open circles indicate the typical eddy viscosity used, in practice, by actual NLOM simulations at that grid resolution [*Hurlburt and Hogan*, 2000; *Hurlburt et al.*, 1996]. At 1/2° there is no point in attempting to resolve eddies, and so A_H = 1500 $m^2\ s^{-1}$ is typically used. At 1/4° degree, A_H = 300 $m^2\ s^{-1}$ is very marginally eddy resolving. Marginally eddy resolving ocean models are often termed eddy permitting, but they commonly use biharmonic rather than Laplacian diffusion [*Gent et al.*, 2002; *Treguier et al.*, 2001] and are therefore outside our scope.

[11] Each further doubling of the model resolution decreases the typical eddy viscosity by a factor of three (Figure 1). All the typical viscosities are in the "converging" zone, because we are attempting to "converge" on the actual behavior of the ocean rather than the continuum limit of the finite difference model equations. As grid resolution increases, the typical viscosity value used in the NLOM gets closer to the converged zone.

[12] In practice, for regions similar to that used in this paper, a A_H value would be typically chosen in the converging category.

4. Eddy Viscosity Variations With Grid Resolution

[13] To illustrate the results given in Figure 1, we examine mean sea surface height (SSH), a good indicator of mean flow, and eddy kinetic energy (EKE) from the first model layer, a good indicator of flow instabilities. The

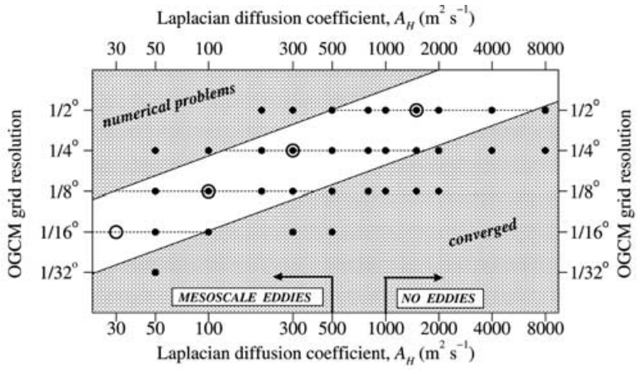

Figure 1. Laplacian diffusion coefficient (also known as horizontal eddy viscosity) values with respect to grid resolutions used for the OGCM (NLOM) simulations. Solid dots indicate experiments run to statistical equilibrium, and open circles indicate the "typical" eddy viscosity that would be used at the given grid resolution. Note that the Laplacian diffusion used in our sequence of simulations is $A_H\ \nabla^2(h\ \vec{V})$ as explained in the text.

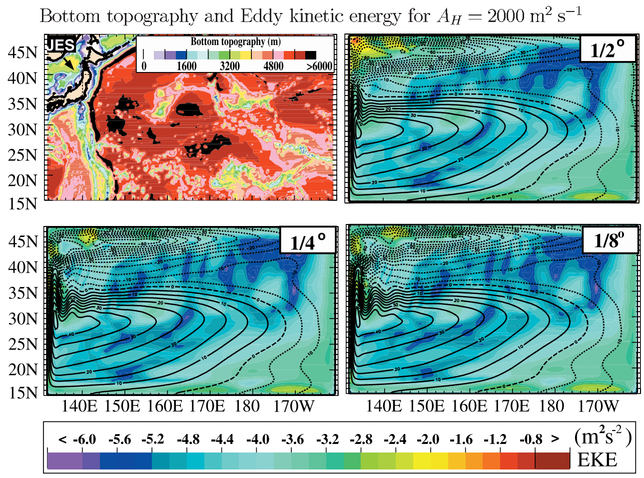

Figure 2. Bottom topography (upper left panel) for the ocean model domain, and surface layer mean Eddy kinetic energy (EKE) per unit mass fields calculated over three years from the model simulations configured at grid resolutions of 1/2°, 1/4° and 1/8°. Mean sea surface height (SSH) is overlain on EKE. The horizontal eddy viscosity coefficient (A_H) value of 2000 $m^2\ s^{-1}$ is used in the ocean model simulations. Note that EKE is given as "log scale" in the color bar. SSH is plotted using solid lines (in black) with a contour interval of 5 cm, and dotted SSH contours represent negative values, i.e., less than the domain average. Long dashed contour lines are used for zero SSH. Japan/East Sea (JES) has been converted into a plateau for the model simulations.

means of SSH and EKE are calculated over three years. At least a three-year mean was needed because the finer resolution NLOM simulations (e.g., finer than 1/2°) are nondeterministic and flow instabilities have a major impact on the results. The simulated SSH and EKE fields are then compared with those from the highest grid resolution available at the same eddy viscosity.

[14] Figure 2 shows EKE and SSH at three different model grid resolutions when A_H is set to 2000 $m^2\ s^{-1}$. The 1/4° and 1/8° results are virtually identical, demonstrating convergence at 1/4°. The 1/2° result is similar, but not identical, to those at higher resolution; indicating that this is in the converging category. The very low values for EKE are typical of non-eddy-resolving simulations. When A_H = 500 $m^2\ s^{-1}$ is used in the simulations at four grid resolutions (Figure 3a), the results from the 1/8° and 1/16° model are nearly same, indicating an almost converged solution at 1/8°. Although comparison with the 1/8° simulation indicates that the 1/4° simulation has serious numerical problems near the western boundary, we place it in the converging category (with the preceding caveat) because 1/4° simulations with realistic geometry for this region do not exhibit such aberrant western boundary current behavior, just qualitative and quantitative inaccuracy. The simulation from the 1/2° model is completely different again, showing the narrow zonal jets that are often associated with finite difference truncation error and is therefore placed in the numerical problems category. However, simulations performed with A_H = 50 $m^2\ s^{-1}$ clearly demonstrate that the 1/16° and 1/32° results are similar but not identical (Figure 3b).

[15] No 1/64° results are available due to computational cost, but convergence is assumed at 1/32°. Despite the severe western boundary current overshoot in the 1/8° simulation, we place it in the converging category for the same reason as the 1/4° simulation with A_H = 500 $m^2\ s^{-1}$ [see *Hurlburt et al.*, 1996, Figure 7 and Plate 8]. The 1/4° result is completely different again, showing almost no penetration of the primary jet into the basin and a western boundary overshoot that continues along the northern boundary and part of the eastern boundary. It is clearly in the numerical problems category.

[16] Comparing the last panels from Figures 2, 3a, and 3b demonstrates the large impact that changing the eddy viscosity can have on the solution. Suppose the 1/32° A_H = 50 $m^2\ s^{-1}$ solution represents the real state of an actual ocean region. If we insist on running simulations that are fully converged, then at 1/8° the best we can do is A_H = 500 $m^2\ s^{-1}$ which is very far from an accurate representation. If simulations are allowed in the converging category, then the 1/8° A_H = 50 $m^2\ s^{-1}$ is one possibility which is significantly closer than A_H = 500 $m^2\ s^{-1}$ to the desired result. In practice, we would typically use A_H = 100 $m^2\ s^{-1}$ at 1/8° (not shown), because it is a safer distance from the numerical problem zone.

5. Conclusions

[17] The most significant results from this paper are: (1) the ocean model resolution required for convergence

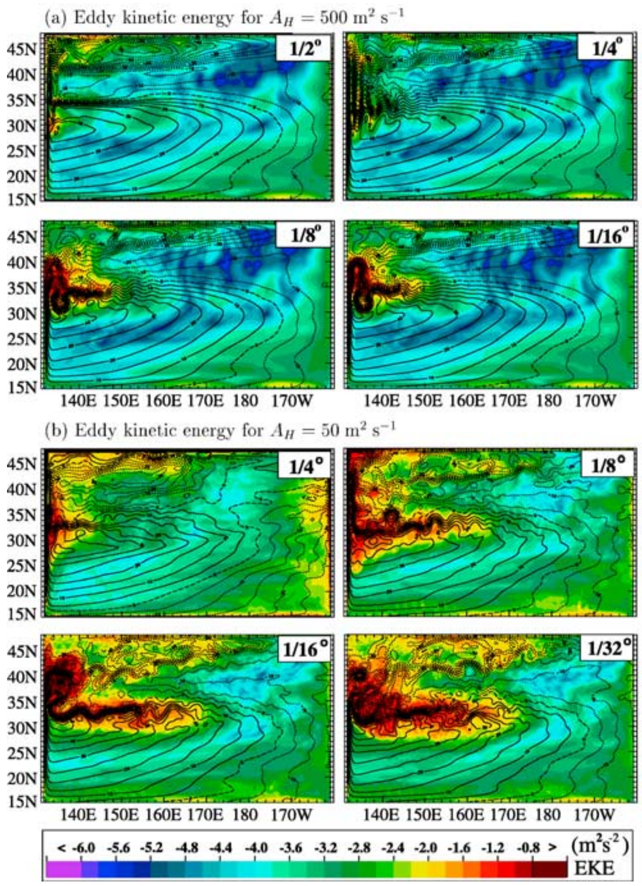

Figure 3. The same as Figure 2 but the ocean model uses (a) 500 $m^2\ s^{-1}$ and (b) 50 $m^2\ s^{-1}$ at grid resolutions ranging from 1/2° to 1/32°.

of Laplacian diffusion is proportional to the square of the grid spacing; (2) it is not safe to use the lowest possible eddy viscosity because there is high risk for grossly inaccurate solutions; (3) for a given horizontal model resolution the most realistic ocean simulations are obtained when the eddy viscosity is smaller than required for convergence to the continuum limit but large enough to avoid gross finite difference truncation error effects (the usual practice of ocean modelers), and (4) the typical eddy viscosity gets closer to the converged value as resolution is increased (because the practical eddy viscosity is reduced by a factor of three, not four, for each doubling of grid resolution).

[18] This study is for a particular model in a particular idealized region. However, our experience with the ocean model used in this paper suggests that although the actual values of A_H used can be region dependent (based on maximum current speeds), the results are insensitive to vertical resolution, and that the same general patterns illustrated here apply to all ocean regions. NLOM has fewer sources of diffusion than some other ocean models, but these results are robust enough to suggest that they may apply to most if not all ocean models that combine Laplacian diffusion with second order finite differences in space.

[19] **Acknowledgments.** This work is a contribution of the projects, Kuroshio Extension Regional Experiment and Global Remote Littoral Forcing via Deep Water Pathways. Projects were funded by the Office of Naval Research (ONR) under program element 601153N. This is contribution NRL/JA/7320/05/5105 and has been approved for public release. Authors would like to thank two anonymous reviewers for their helpful suggestions.

References

Berloff, P. S., and J. C. McWilliams (2002), Material transport in oceanic gyres, part II: Hierarchy of stochastic models, *J. Phys. Oceanogr.*, *32*, 797–830.

Gent, P. R., A. P. Craig, C. M. Bitz, and J. W. Weatherly (2002), Parameterization improvements in an eddy-permitting ocean model for climate, *J. Clim.*, *15*, 1447–1459.

Griffies, S. M., R. C. Pacanowski, and R. W. Hallberg (2000), Spurious diapycnal mixing associated with advection in a *z*-coordinate ocean model, *Mon. Weather Rev.*, *128*, 538–564.

Hellerman, S., and M. Rosenstein (1983), Normal monthly wind stress over the world ocean with error estimates, *J. Phys. Oceanogr.*, *13*, 1093–1104.

Hogan, P. J., and H. E. Hurlburt (2000), Impact of upper ocean-topographical coupling and isopycnal outcropping in Japan/East Sea models with 1/8° to 1/64° resolution, *J. Phys. Oceanogr.*, *30*, 2535–2561.

Hurlburt, H. E., and P. J. Hogan (2000), Impact of 1/8° to 1/64° resolution on Gulf Stream model-data comparisons in basin-scale subtropical Atlantic Ocean models, *Dyn. Atmos. Oceans*, *32*, 283–329.

Hurlburt, H. E., A. J. Wallcraft, W. J. Schmitz Jr., P. J. Hogan, and E. J. Metzger (1996), Dynamics of the Kuroshio/Oyashio current system using eddy-resolving models of the North Pacific Ocean, *J. Geophys. Res.*, *101*, 941–976.

Kara, A. B., A. J. Wallcraft, and H. E. Hurlburt (2003), Climatological SST and MLD simulations from NLOM with an embedded mixed layer, *J. Atmos. Oceanic Technol.*, *20*, 1616–1632.

Kara, A. B., H. E. Hurlburt, P. A. Rochford, and J. J. O'Brien (2004), The impact of water turbidity on the interannual sea surface temperature simulations in a layered global ocean model, *J. Phys. Oceanogr.*, *34*, 345–359.

Killworth, P. D. (1997), On the parameterization of eddy transfer, part I: Theory, *J. Mar. Res.*, *55*, 1171–1197.

National Oceanic and Atmospheric Administration (1988), Digital relief of the surface of the Earth, *Data Announce. 88-MGG-02*, Natl. Geophys. Data Cent., Boulder, Colo.

Schmitz, W. J., Jr., and J. D. Thompson (1993), On the effects of horizontal resolution in a limited-area model of the Gulf Stream system, *J. Phys. Oceanogr.*, *23*, 1001–1007.

Shchepetkin, A. F., and J. J. O'Brien (1996), A physically consistent formulation of lateral friction in shallow-water equation ocean models, *J. Geophys. Res.*, *124*, 1285–1300.

Smith, R. D., and P. R. Gent (2004), Anisotropic Gent-McWilliams parameterization for ocean models, *J. Phys. Oceanogr.*, *34*, 2541–2564.

Treguier, A. M., B. Barnier, A. P. de Miranda, J. M. Molines, N. Grima, M. Imbard, G. Madec, C. Messager, T. Reynaud, and S. Michel (2001), An eddy permitting model of the Atlantic Circulation: Evaluating open boundary conditions, *J. Geophys. Res.*, *106*, 22,115–22,129.

Wallcraft, A. J., A. B. Kara, H. E. Hurlburt, and P. A. Rochford (2003), The NRL Layered Global Ocean Model (NLOM) with an embedded mixed layer submodel: Formulation and tuning, *J. Atmos. Oceanic Technol.*, *20*, 1601–1615.

H. E. Hurlburt, A. B. Kara, and A. J. Wallcraft, Naval Research Laboratory, Oceanography Division, Code 7320, Bldg. 1009, Stennis Space Center, MS 39529, USA. (hurlburt@nrlssc.navy.mil; kara@nrlssc.navy.mil; wallcraft@nrlssc.navy.mil)