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Use of phase-resolving wave models in bathymetry deduction

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ABSTRACT:

We invert for the bathymetry under random shoaling waves using two different inverse techniques: direct inversion using Levenberg-Marquardt method and a variational inverse method. For both methods, we parameterize the bathymetry using the Bruun-Dean's profile. For the direct inversion, we first show that bathymetric parameters can be obtained with synthetic random wave data. Next, we perform the inversion to show that the bathymetry is obtained exactly using both the adjoint model and direct inversion methods.

1 Introduction

The bottom topography strongly influences the wave field. Hence, the knowledge of the variations of the bathymetry is very important for the proper understanding of the complex interactions between waves and currents. Typically, in nearshore wave problems, water depth is assumed to be known. The nonlinear wave propagation over known varying bathymetries can be determined by nonlinear time dependent Boussinesq model's such as FUNWAVE (Wei *et al.*, 1995). The wave response due to changes in depth is thus determined. This we define as the forward problem. Here, we are interested in the inverse problem. If some information of the dynamics and/or kinematics is known, is it possible to obtain the bathymetry ?

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The bathymetric inversion can be achieved in many ways. For all inversion methods, a model is necessary. This model can be a phase-averaged or a phase-resolving model. In this paper, we will restrict our attention to phase-resolving numerical models. We use two different approaches to obtaining the bathymetry. The first is to use a direct inversion method using a nonlinear least squares technique. In this method, the model used is assumed to be perfect. i.e, if we have exact initial and boundary conditions there will be no error in the model. The second method is the variational inverse method. In this method, a positive definite function is minimized subject to constraints imposed by the model and the data. In this method, it is possible to weight the data and model so that the model results are assumed to be imperfect.

Many people have conducted research using the direct inversion method. Kennedy *et al.* (1999) used a fully nonlinear time-dependent Boussinesq model as a tool for the inversion. They assume that time lagged spatial maps of both synthetic surface wave height and orbital velocities are available as inputs to their algorithm for inverting depth. They predict water-depths for various wave minimization conditions reliably. Misra *et al.* (2000) using only orbital velocities, developed an inversion method to calculate the bathymetry and surface elevation. Recently, Putrevu *et al.* (2000) deduced the nearshore bathymetry and currents using remote measurements of the edge wave kinematics. Narayanan and Kaihatu (2000) (henceforth, NK00) studied the sensitivity of the collected data with respect to the bathymetric parameters. Using a Kortweg-de Vries (KdV) wave model they showed that parameterizing the water-depth using the Bruun/Dean's profile (Bruun, 1954; Dean, 1977) or an exponential profile (Bodge, 1992) are most suitable for inversion. However, they restricted their attention to regular waves. In this paper, we extend the analysis to include irregular waves.

One question that we attempt to answer in this paper is: Can the bathymetry be obtained when random surface wave information is given ? In an approach similar to NK00, we develop a sensitivity matrix to determine if the bathymetric parameters can be estimated for random surface data. Next, we invert for the bathymetry and discuss the results. Another question is: Can the inverse bathymetry be obtained using the forward and adjoint model ? Here, we develop the Euler-Lagrange equations and solve the forward and backward equation subject to the of the cost function to obtain the bathymetry.

For both inversion techniques, we parameterize the bathymetry using the Bruun-Dean's profile (Bruun, 1954; Dean, 1977). The water-depth is described as $h(x) = Ax^m$. Thus, there are undetermined two parameters: A and m.

2 Direct Inversion using KdV model

In this section, we formulate the problem, discuss the methodology and show results using the direct inversion technique.

2.1 Data and Model

Depth inversion involves collecting data in terms of time-series or snapshots of surface elevations and/or particle velocities and then deducing the depth by establishing a relationship between the collected data and depth. In the present context, we will define data as the free-surface data collected in terms of time-series or free surface imagery. We need to obtain the bathymetry from the data set that we have access to. The bathymetry is described by model parameters. We assume there is a specific method (usually a mathematical theory or model) for relating the model parameters to the data. In this work, we use the KdV equation as the model that connects the data and model parameters. We assume that the physics of shoaling is completely known and well described by the KdV equation. Since the emphasis of this work is not to obtain the best model for wave shoaling but to understand the dependence of the the model parameters to the data, this approach is justified.

In order to estimate if the parameters are identifiable, we develop a matrix R which contain the gradients of the data with respect to the parameters (see NK00 for more details). Here η is the collected data and A and m are the parameters.

$$R = \begin{pmatrix} \frac{\partial \eta_1}{\partial A} & \frac{\partial \eta_1}{\partial m} \\ \frac{\partial \eta_2}{\partial A} & \frac{\partial \eta_2}{\partial m} \\ \frac{\partial \eta_3}{\partial A} & \frac{\partial \eta_3}{\partial m} \\ \vdots & \vdots \\ \frac{\partial \eta_n}{\partial A} & \frac{\partial \eta_n}{\partial m} \end{pmatrix}$$

We can also form a $R^T R$ matrix (where R^T is the transpose of the R matrix) whose diagonal elements are $\sum_{i=1}^{n}(\frac{\partial \eta_i}{\partial A})^2$ and $\sum_{i=1}^{n}(\frac{\partial \eta_i}{\partial m})^2$. These two terms represent the sensitivity of A and m with respect to the data. If the two diagonal elements are within three orders of magnitude, then the parameters are said to be identifiable. Otherwise, the parameters cannot be identified. Note that even if the parameters are identifiable, it is not necessarily true that inversion will be possible. There may be problems associated with non-uniqueness and instability which may not allow for the estimation of the bathymetry.

Now, the bathymetry is parameterized using the equation $h(x) = A(1 - (x/x_{max}))^m$. The two parameters that are to be identified are A and m. This equation is similar to the Bruun/Dean's profile but is normalized by x_{max}. The

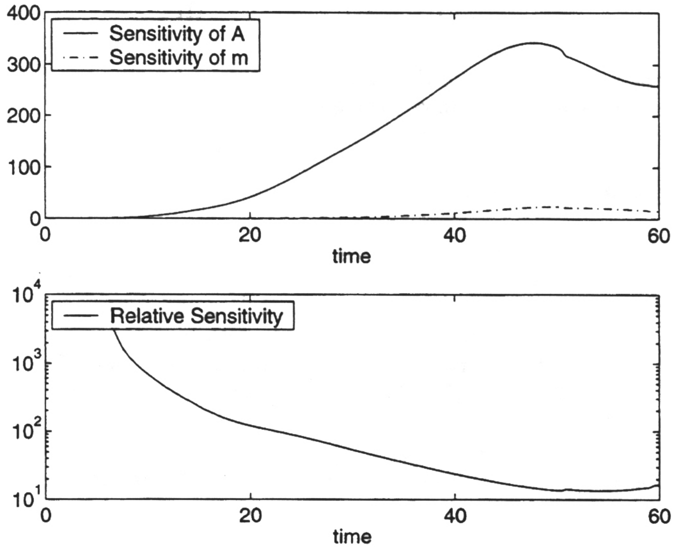

Figure 1: The top figure shows the sensitivity for the Bruun/Dean's profile as a function of times at which data is collected. The bottom panel shows the relative sensitivity for the Bruun/Dean's profile as a function of times at which data is collected.

numerical domain is 35 m long with the wavemaker at $x = 0$. There are 3500 points in the domain, $\Delta x = 0.01$ m and $\Delta t = 0.01$ s. At $t = 0$, $\eta = 0$ for all x. At $x = 0$, we generate random waves using the TMA spectrum (Hughes, 1984) with the narrowness constant $\gamma = 3.3$ (Cox *et al.*, 1991). We collect the free-surface data between times $t = 5$ s to $t = 24$ s with an interval of 0.1 s. Now the model is run again changing changing A and m one at a time to obtain the gradients of the data with respect to the two parameters. Both the R and $R^T\,R$ matrices are obtained. For each of these times, we run the KdV model to obtain the sensitivity. Figure 1 (top) shows the sensitivity for A and m as a function of the times at which the data was collected. The relative sensitivities (defined as ratio of the sensitivity of A to the sensitivity of m) are also plotted as a function of time (bottom of figure 1). The figures shows that for data collected after $t = 15$ s, the ratio of the sensitivities for both A and m are within two orders of magnitude. Hence, the necessary condition for the parameter identifiability is satisfied. Therefore, for data collected at all times greater than $t = 15$ s, it is highly likely that the two parameters A and m can be identified.

2.2 Results

Since we have already established that the bathymetric parameters can be estimated using the random wave data, the next step is to invert for the bathymetry. The procedure for solving the inverse problem is as follows. A particular depth profile is chosen ($A = 0.4$ and $m = 0.66$). The KdV model is run with the boundary conditions and the chosen bathymetry. Data is obtained in the form of snapshots at different times ($t = 16$, 20 and 24 s). With a different starting point ($A = 0.8$; $m = 0.8$) for the depth, the model is run again and data are collected (We call this the observed data.). The aim now is to minimize the difference between the true data and the observed data to obtain reasonable estimates for the parameters. Since this is a nonlinear problem, the simplest technique that we can use is the nonlinear least-squares method with damping (Levenberg-Marquardt technique, NK00).

The model converges to the true solution within 15 iterations. Figure 2 shows the data collected at the three different times in the top panel. The middle panel shows the bathymetry at different iterations. The bottom panel shows the convergence plot as a function of A and m. We attempted other starting points and obtained similar results. As in the case of regular waves, there are regions of divergence; however, the primary outcome of this analysis is that the bathymetry can be determined with random wave information.

3 Adjoint model

In the adjoint inverse problem , we look for the minimum of a cost function J, while the dynamic equations $E(\eta) = 0$ describing the temporal evolution of the models state variable (η) act as a set of strong constraints, i.e. they are fulfilled exactly. The assimilation occurs between time periods t_1 to t_2. Within the assimilation period the evolution of the model variables is fully controlled by the (discrete) model physics and the initial and boundary conditions. Variables describing these conditions are commonly denoted control variables u.

We choose control variables A and m (denoted as u) such that the least square distance between model and data, cost function J, is minimized. That is solved by applying a iterative procedure. First the gradient of the cost function with respect to the control variables is calculated. This gradient is then passed to a general unconstrained minimization algorithm such as quasi-Newton or conjugate gradient, which will iteratively give better and better estimates for the control variables u. This optimization procedure is called the adjoint method, because the adjoint model equations offer a sophisticated but relatively cheap way to calculate the gradient .

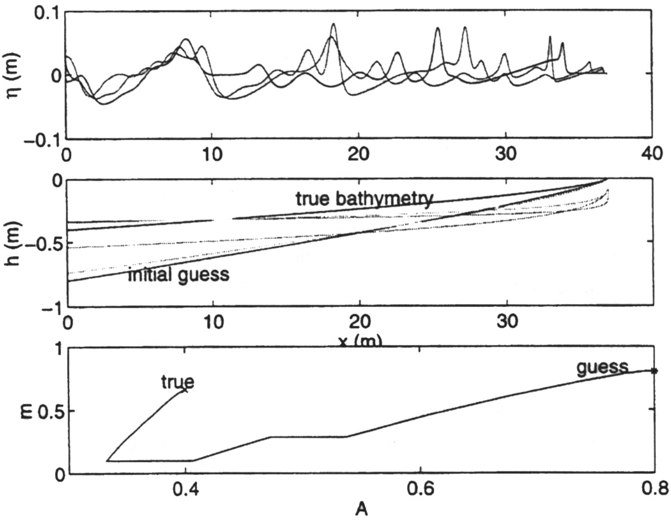

Figure 2: The data collected at the three different times in shown in the top panel. The middle panel shows the bathymetry at different stages. The bottom panel shows the convergence plot as a function of A and m.

Following the formalism of the variational method the Lagrange function

$$L = J + \sum_{x=1}^{L} \sum_{t=1}^{T} \lambda E(\eta, x, t) dt\, dx \tag{1}$$

is introduced. Hence the problem, which is a constrained minimization in the space of model variables, is transformed into a unconstrained problem in the space of control variables u. The Euler-Lagrange equations describe a stationary point (saddle point) of L. Partial differentiation with respect to the Lagrange multipliers , also denoted as adjoint variables, returns the model equations. This can be expressed as

$$L_\lambda = E(\eta) = 0 \tag{2}$$

The differentiation with respect to the model variable (η) yields (after partial integration) the adjoint model equation, which describe the temporal evolution of the Lagrange multipliers. The partial integration also yields the boundary conditions for λ.

$$L_\eta = 0 \tag{3}$$

The third condition

$$L_u = 0 \tag{4}$$

ensures that the optimal choice of the control variables u has been found.

Our Euler-Lagrange system of equations is:

- Model Equation

$$\eta_t + (c_0\eta)_x + [\frac{3c_0\eta^2}{4h} - \frac{h^2\eta_{xt}}{6}]_x = 0 \tag{5}$$

- Euler-Lagrange Equation

$$\lambda_t + (c_0\lambda)_x + \frac{3c_0}{2}\eta(\lambda/h)_x - (\frac{h^2\lambda_{xt}}{6})_x = K_\eta(\eta - \eta') \tag{6}$$

Here η' is the data at any particular iteration. The third condition is too complex and hence we did not use it. Instead, we used a simpler criterion for the convergence. We know that the minimum occurs when $(\eta - \eta')$ is minimal. This also corresponds to $\lambda(x,t)$ being small. Hence, we used the criterion to obtain the best solution.

$$\sum_{x=1}^{L}\sum_{t=t1}^{t2} \lambda_{x,t} = 0 \tag{7}$$

Of these three equations (5,6,7) only the model equations (5) can be solved by integrating them from t_1 to t_2. The second (i.e. the adjoint equations) is integrated backward in time from t_2 to t_1. In general, the first guess for A and m will not be optimal.

We perform the inverse for the model and adjoint equations described above. The methodology is as follows. The KdV model is run with $A = 0.4$ and $m = 0.66$. Snapshots of the free-surface (η) is stored at times $t = 16$, 20 and 24 s. We choose $A = 0.8$ and $m = 0.8$, run the model again and collect data (η'). The adjoint equations are driven by the mismatch $(\eta - \eta')$. We then use the criterion in equation 7 and solve the model and adjoint equations iteratively to converge to the true solution. The true solution is obtained within 20 iterations. We also run the forward-backward equations for other starting points (i.e., different A and m). Figure 3 shows the convergence plot as a function of the control variables A and m for different model runs. The figure shows regions of convergence (* symbols) and divergence (+ symbols). So, clearly the solution depends on the initial guess for the control variables. The primary advantage of the adjoint model approach is that we can assign weights to the data and model based on our certainty.

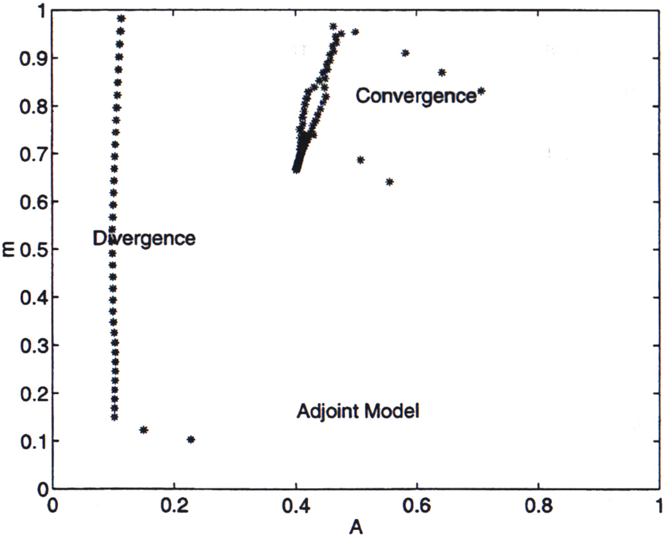

Figure 3: The figure shows the convergence plot as a function of the control variables A and m. The figure shows regions of convergence (* symbols) and divergence (+ symbols).

4 Discussion

Narayanan and Kaihatu (2000) performed a feasibility study to show that bathymetry can be determined using inverse methods. They also discussed the conditions under which the bathymetry can be determined. However, their analysis was restricted to regular waves using direct inversion techniques. In this paper, we extend their analysis by studying irregular waves using two different techniques:

direct inversion using Levenberg-Marquardt technique and the adjoint model assimilation technique. We show that inversion for the bathymetry is possible for irregular waves using both the techniques.

We have an excellent idea of where and how to collect data for inversion to be possible. This can be determined using sensitivity analysis discussed in both this work and NK00. The next step is to use real data obtained from the field to obtain the bathymetry. All the analysis conducted so far have used synthetic data. Due to the strongly nonlinear nature of the problem, it is difficult to use the inversion techniques directly to obtain the bathymetry using real data. Other methods (e.g., simulated annealing, neural networks, etc.) may prove to be more fruitful.

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